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An Analysis on the Role of Fixed Point Theory in Complete Cone Metric Spaces: A Review

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Abstract – We talk about the recently presented idea of cone metric spaces. We additionally examine the fixed point presence results of contractive mappings characterized on such metric spaces. Specifically, we demonstrate that a large portion of the new results are just duplicates of the established ones. In this paper we set up some Fixed Point theorems by altering distances in a complete cone metric space with supposition that the cone is normal, likewise we present MS-Altering function.

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INTRODUCTION

Fixed point theorems give conditions under which maps (single or multivalued) have arrangements. The theory itself is a wonderful blend of investigation, topology, and geometry. In the course of the most recent 80 years or so the theory of fixed points has been uncovered as an exceptionally amazing and significant instrument in the investigation of nonlinear marvels. Specifically, fixed point systems have been connected in such assorted fields as Biology, Chemistry, Economics, Engineering, Game Theory, and Physics. Fixed point theory assumes a significant job in functional examination; guess theory, differential conditions and applications, for example, limit esteem issues and so on. As of late, progressively fixed point results in cone metric spaces showed up. Topological inquiries in cone metric spaces were contemplated where it was demonstrated that each cone metric space is first countable topological space. Thus, progression is proportionate to consecutive congruity and smallness is identical to successive minimization. It merits referencing the spearheading work of Quilliot who presented the idea of generalized metric spaces. His methodology was effective and utilized by many. It is our conviction that cone metric spaces are a unique instance of generalized metric spaces. In this work, we present a metric sort structure in cone metric spaces and demonstrate that established verifications do convey indistinguishably in these metric spaces. This methodology recommends that any augmentation of realized fixed point result to cone metric spaces is excess. Besides the fundamental Banach space and the related cone subset are a bit much.

Guang and Zhang (2007) as of late presented the idea of cone metric spaces and built up some fixed point theorems for contractive sort mappings in a normal cone metric space. Hence, some different creators contemplated the presence of points of fortuitous event and normal fixed points of mappings fulfilling a contractive kind condition in cone metric spaces. Thereafter, Rezapour and Hamlbarani (2008) demonstrated fixed point theorems of contractive kind mappings by precluding the presumption of normality in cone metric spaces.

In this paper we get points of fortuitous event and regular fixed points for three self mappings fulfilling Jungck (1976) type contractive condition with the supposition of normality in cone metric spaces.

First we review Jungck's theorem :

THEOREM 1 : Let (X, ρ) be a complete metric space. Give f a chance to be a constant self-map on X and g be any self-map on X that drives with f. Further given f and g a chance to fulfill $g(X) \subseteq f(X)$ and there exists a consistent $\lambda \in (0, 1)$ such that for each

$$x, y \in X, \quad \rho(gx, gy) \leq \lambda \rho(fx, fy).$$

At that point f and g have a novel regular fixed point.

Sessa (1982) generalized the idea of commuting mappings by calling self mappings f,g on a metric space X, pitifully commuting if and just if $d(fgx, gfx) \leq d(fx, gx)$ for every one of the $x \in X$. Clearly commuting mappings are weakly commuting yet banter isn't valid all in all. Subsequently, numerous authors obtained decent fixed point theorems by utilizing this idea.

Along these lines Pant (1994) presented some less prohibitive ideas of good mappings and R-pitifully commuting mappings separately. Later on, it has been seen that perfect mappings and R-feebly commuting mappings drive at their fortuitous event point. Jungck and Rhoades (1998), at that point characterized a couple (f,g) of self mappings to be pitifully good on the off chance that they drive at their occurrence point (i.e. fgx = gfx at whatever point fx = gx).

DEFINITION 1 : Let f and T act naturally maps of a nonempty set X. In the event that there exists $x \in X$ such that fx = Tx then x is known as an incident point of f and T, while y = fx = Tx is known as a point of occurrence of f and T.

MAIN RESULTS-

In this segment, we demonstrate two fundamental results and get the results of Azam. Arshad and Beg (2008) as conclusions.

We begin with a lemma, which will be required in the continuation.

LEMMA 1: Let X be a non-void set and the mappings $S,T,f: X \to X$ have a one of a kind point of happenstance v inX. On the off chance that (S,f) and (T,f) are pitifully perfect, at that point S, T and f have an extraordinary regular fixed point.

THEOREM 2 : Let (X, d) be a cone metric space and the mappings $S, T, f: X \to X$ satisfy: $d(Sx, Ty) \leq \lambda\, d(fx, fy)$ for each of the $x, y \in X$ where $0 \leq \lambda < 1$. In the event that $S(X) \cup T(X) \subseteq f(X)$ and f(X) is a complete subspace of X , then S ,T and f have a novel point of fortuitous event . Besides on the off chance that (S,f) and (T,f) are pitifully perfect, at that point S, T and f have a special basic fixed point.

In this theorem, in the event that we take x = y, at that point we will get S = T. Subsequently this theorem might be taken as a typical fixed point theorem for two maps S and f.

FIXED POINT THEOREMS IN COMPLETE CONE METRIC SPACES OVER BANACH ALGEBRAS

Huang and Zhang (2007) presented cone metric spaces which are speculations of metric spaces, and they stretched out Banach's constriction principle to such spaces, whereafter numerous creators contemplated fixed point theorems in cone metric spaces. As of late, Liu and Xu (2013) presented the thought of cone metric spaces over Banach algebras, which is a change of the idea of cone metric spaces over genuine Banach spaces, and demonstrated the presence of fixed points for mappings characterized on such spaces, and they gave an example that fixed point results in metric spaces and in cone metric spaces are not equal.

In all respects as of late, Chandok et al. (2016) presented the idea of TAC-contractive mappings by utilizing the ideas of C-class functions and cyclic (α, β)- acceptable mappings and set up relating fixed point theorems in metric spaces.

In the paper, we present the thoughts of C-class functions and cyclic (α, β)- allowable mappings in Banach algebras. By utilizing such concepts,we present a newcontraction. We acquire another fixed point theorem and give an example to show fundamental result. At long last, we give utilizations of our principle result to cyclic mappings and feeble constriction type mappings in cone metric spaces over Banach algebras.

A Banach space $\mathcal{A}$ is known as a (genuine) Banach algebra (with unit) if there exists multiplication $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ that has the followings properties:

for all $x, y, z \in \mathcal{A}, \alpha \in \mathbb{R}$,

1. (xy)z = x(yz);
2. x(y + z) = xy + xz and (x + y)z = xz + yz;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. there exists $e \in \mathcal{A}$ such that xe = ex = x;
5. $\|e\| = 1$;
6. $\|xy\| \leq \|x\|\|y\|$.

An element $x \in \mathcal{A}$ is called invertible if there exists $x^{-1} \in \mathcal{A}$ such that $xx^{-1} = x^{-1}x = e$.

PROPOSITION :. Let $\mathcal{A}$ be a Banach algebra, and let $x \in \mathcal{A}$. If the spectral radius $\rho(x)$ of x is less than 1, i.e.,

$$\rho(x) = \lim_{n \longrightarrow \infty} \|x^n\|^{1/n} = \inf_{n \geq 1} \|x^n\|^{1/n} < 1,$$

Then $e - x \in G(\mathcal{A})$, where $G(\mathcal{A})$ is the set of all invertible elements of si and

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

REMARK : Let $\mathcal{A}$ be a Banach algebra. Then the following are satisfied:

1. for all $x \in \mathcal{A}, \rho(x) \leq \|x\|$;

2. If the condition $\rho(x) < 1$ is replaced by $\|x\| < 1$ in Proposition 1, then the conclusion holds.

Consistent with Liu and Xu (2013), the following definitions will be needed in the sequel. Let $\mathscr{A}$ be a Banach algebra. A subset P o f $\mathscr{A}$ is called cone if the following conditions are satisfied:

(1) P is a nonempty and closed subset of si and {0, e} c P;

(2) $ax+by \in P,$ whenever $x, y \in P$ and $a,b \in \mathbb{R}(a,b \geq 0);$

(3) $P^2 = PP \subset P;$

(4) $P \cap (-P) = \{0\}.$

FIXED POINT THEOREMS IN CONE METRIC SPACES BY ALTERING DISTANCES

Since the Banach Contraction Principles, a few kinds of speculation withdrawal Mappings on metric spaces have showed up. One such technique for speculation is altering the distances. Delbosco and Skof have set up Fixed Point Theorems for self maps of complete metric spaces by altering the distances between the points with the utilization of a positive genuine esteemed function.

Huang and Zhag (2007) presented the idea of cone metric space by supplanting the arrangement of genuine numbers by an arranged Banach space and got some fixed point results. As of late Asadi and Soleimani demonstrated some Fixed Point results on cone metric space by utilizing altering separation function and the (ID) Property of somewhat requested cone metric space.

We are giving some new results by presenting a vector esteemed function (Malhotra-Shukla altering function) in cone metric spaces which has similitude with altering function. It turns into the speculation of Altering Function in perspective on cone utilized instead of positive genuine numbers, just as the imperatives utilized for self map of cone metric spaces.

PRELIMINARIES-

DEFINITION 1 ; Let E be a genuine Banach space and P be a subset of E. P is known as a cone if

(a) P is a closed, nonempty and $P \neq \{0\};$

(b) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P;$

(c) $x \in P and -x \in P \Rightarrow x = 0.$

Given a cone $P \subseteq E,$ we define a partial ordering "$\leq$" in E by $x \leq y$ if $y - x \in P.$ We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x \ll y$ to denote $y - x \in P^0$, where P^0 stands for the interior of P. We assume that $P^0 \neq \phi.$ P is called normal if for some $M > 0$, for all $x, y \in X, 0 \leq x \leq y$ implies $\|x\| \leq M\|y\|.$

PROPOSITION 2: Let P be a cone in a real Banach space E. If for $a \in P$ and $a \leq ka,$ for some $k \in [0,1)$ then a=0.

PROPOSITION 3: Let P be a cone in a real Banach space E. If for $a \in E$ and $a \ll c,$ for all $c \in P^0$ then a = 0.

PROPOSITION 4 ; Let P be a cone in a real Banach space E. If a,b E E and $a \ll b$ and $b \ll c$ then $a \ll c$ and if $a \leq b$ and $b \ll c$ then $a \ll c.$

DEFINITION 5: Let X be a nonempty set and E be a real Banach space. Suppose that the mapping $d : X \times X \to E$ satisfies

(a) $0 \leq d(x, y),$ for all $x, y \in X$ and $d(x, y) = 0,$ if and only if x = y;

(b) $d(x, y) = d(y, x)$, for all $x, y \in X;$

(c) $d(x, y) \leq d(x, z) + d(z, y),$ for all $x, y, z \in X.$

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

EXAMPLE 6: Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$ and $d : X \to X$ such that $d(x, y) = (|x - y|, \alpha|x - y|),$ where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space. Henceforth unless otherwise indicated, P is a normal cone in real Banach space E and "$\leq$" is partial ordering with respect to P.

FIXED POINTS FOR CONTRACTIVE TYPE MAPPINGS IN S-CONE METRIC SPACES

The motivation behind this paper is to demonstrate some basic fixed point results in cone metric spaces. Aage and Salunke (2009) demonstrated two theorems on cone metric spaces. We demonstrate that they are trifling. We likewise give an example with that impact and make reasonable alterations in the speculation of the theorems to make the end viable. For this reason we present the idea of S-cone metric spaces and demonstrate a theorem on fixed points of a self map on a S-cone metric space. Supporting example to the theorem is additionally given.

Huang and Zhang (2007) presented the idea of cone metric spaces and some fixed point theorems for contractive mappings were demonstrated in these spaces.The results in Huang and Zhang (2007) were generalized by Sh.Rezapour and R.Hamlbarani in Rezapour Sh and Hamlbarani R (2008). Thusly Abbas and Jungck (2008), and

Ismat Beg (2009) have researched some regular fixed point theorems for various sorts of contractive mappings in cone metric spaces. For completeness purpose, We begin with fundamental definitions on cone metric spaces.

DEFINITION 1: Let E be a genuine Banach space and P a subset of E. P is known as a cone if

(I) P is shut, non-void and $P \neq \{0\}$

(ii) $ax + by \in P \ \forall \ x, y \in P$ and non-negative genuine numbers an and b.

(iii) $P \cap (-P) = \{0\}$.

DEFINITION 2 : We characterize a fractional requesting $\leq$ on E regarding P and $P \subset E$ by $x \leq y$ if and just if $y - x \in P$. We will compose $x << y$ if $y - x \in int\, P$, $int\, P$ denotes the interior of P.

We signify by $\|\cdot\|$ the standard on E. The cone P is called normal if there is a number $K > 0$ with the end goal that for every one of the $x, y \in E, 0 \leq x \leq y$ infers $\|x\| \leq K \|y\|$ ...(1)

The least positive number K fulfilling (1) is known as the normal steady of P.

DEFINITION 3 : A cone P is called ordinary if each expanding succession which is limited from above is focalized. That is, on the off chance that $\{x_n\}_{n\geq 1}$ is an arrangement with the end goal that $x_1 \leq x_2 \leq \ldots\ldots \leq y$ for some $y \in E$, at that point there is $x \in E$ to such an extent that $\lim_{n\to\infty} \|x_n - x\| = 0$

We see that a cone P is ordinary if and just if each diminishing succession which is limited from beneath is focalized.

MAIN RESULTS-

In this area we demonstrate that results of Aage and Salunke (2009) are trifling. We likewise present the thought of a S-cone metric space and build up a fixed point theorem in a S-cone metric space pursued by supporting examples.

THEOREM 1 :

Let (X, d) be a complete cone metric space and P a normal cone with normal steady K. Assume that the mappings

$f, g: X \to X$ fulfill:

$$\alpha d(fx, gy) + \beta d(x, fx) + \gamma d(y, gy) \leq \delta d(x, y) \quad ...(2)$$

for every one of the $x, y \in X$ and $\alpha, \beta, \gamma, \delta \geq 0, \beta < \delta, \gamma < \delta, \delta < \alpha.$

At that point f and g have an interesting basic fixed point in X.

In the above theorem, it very well may be effectively seen that f = g by taking x = y.

Further, on the off chance that $\beta > 0$ and $\gamma > 0$ at that point f is the personality map, and on the off chance that $\beta = 0 = \gamma$, at that point this diminishes to Banach constriction principle in cone metric spaces.

Presently we present the idea of a S-cone metric space and get our principle fixed point theorem.

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