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Survey on Fractals

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Abstract* – *From last years, fractals and the study of their dynamics is one of the emerging and attractive areas for mathematicians. A Fractal is a geometric shape each part of which has the same characteristics as the original one. By zooming a fractal, one can find that the patterns and shapes will continue repeating, forever. In this paper, I present a brief survey on fractals.

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1. INTRODUCTION

In mathematics, a dynamical system is a system in which a function describes the time dependence of a point in a geometrical space. It is the study of iteration of maps over a time period [12]. Nonlinear dynamics studies systems governed by equations more complex than the linear, ax + b form. Nonlinear systems, such as the weather or neurons, often appear chaotic, unpredictable and yet their behavior is not random.

Among the major recent developments in understanding the structures of objects found in nature, the notion of fractals occupies an important place. A fractal is a non-regular geometric shape that has the same degree of non-regularity on all scales. In mathematics, a fractal is an object used to describe and simulate naturally occurring objects. A fractal is defined as a fragmented geometric shape which can split into parts that are considered a reduced copy of the whole. Fractal theory is a popular branch of mathematical art. In mathematical visualization, fractals look very beautiful even though they can be created using simple patterns. A fractal is a never ending and infinitely complex pattern that is self-similar across different scales. By zooming a fractal, one can find that the patterns and shapes will continue repeating forever [2, 6].

Fractal:

Fractals are some of the most attractive and most inexplicable geometric shapes. The term “Fractal” was first used by French Mathematician Benoit B. Mandelbrot in 1975 and defined a fractal as a set whose Hausdroff dimension is strictly greater than its topological dimension. It was taken from the Latin word franger, which means broken or fractured. The term was used to describe objects that were not easily fit into traditional geometrical settings. Mathematicians have used fractal geometry to present some interesting complex objects to computer graphics.

According to Pickover [11], the mathematics behind fractals began to take shape in the 17th century when the mathematician and philosopher Gottfried Leibniz considered recursive self-similarity. After two centuries, in 1872 Karl Weierstrass presented the first definition of a function with a graph that would today be considered fractal.

A fractal often has the following features:

(i) It has a fine structure at arbitrarily small scales.

(ii) It is self--similar (at least approximately).

(iii) It is too irregular to be easily described in traditional Euclidean geometry.

(iv) It has a dimension which is non--integer i.e. fraction.

(v) Its fractal dimension is greater than its topological dimension (i.e. the dimension of the space required to “draw” the fractal).

(vi) It has a simple and recursive definition.

Mathematicians have attempted to describe fractal shapes for over one hundred years, but with the processing power and imaging abilities of modern computers, fractals have enjoyed a new popularity because they can be digitally rendered and explored in all of their fascinating beauty.

To create a mathematical fractal, you can start with a simple pattern and repeat it at smaller scales, again and again, forever. In real life, of course, it is impossible to draw fractals with “infinitely small” patterns. Howeverwe can draw shapes which look just like fractals. Using mathematics, we can think about the properties a real fractal would have – and these are very surprising. Following are the results of sets that are commonly referred as fractals.

Julia Set: Fractals are some of the most beautiful and most bizarre geometric shapes. Julia set is an example of such beautiful geometric shapes. Julia sets are certain fractal sets in the complex plane that arise from the dynamics of complex polynomials. Interest in Julia setsand related mathematics began in 1920's with Gaston Julia [2]. Now, fractal theory is incomplete without the presence of Julia sets. Julia sets have been studied for quadratic [5, 8, 2], cubic [4, 9] and higher degree polynomials [3] under Picard orbit [9], which is an example of one-step feedback process. In last few decades many beautiful Julia sets studies using two-step feedback process (Superior orbit) [13], three step feedback process (I – Superior orbit) [13], Noor orbit [1] and SP orbit [10]. Following is the definition of the Julia sets for $Q_c(z) = z^n + c$, where n = 2, 3,...

Definition 1. Let K be the set of points whose orbit is bounded under the function iteration Q(z). Then the set k is called the filled Julia set for the function Q(z). The Julia set Q is the boundary of the filled Julia set K [7].

The following theorem gives the general escape criterion of the Julia sets and further their corollaries refine the escape criterion for the computational purposes using Picard orbit [6, 13], superior orbit, I-Superior orbit, Noor Orbit [1], SP- orbit [10].

Theorem 1. Suppose |z| ≥ |c| > 2, where constant c is in the complex plane. Then we have $|Q_c^n(z)| \to \infty \; as \; n \to \infty$.

Corollary 1. Suppose |c| > 2, then the orbit of 0 escapes to infinity under function iteration Q_c.

Corollary 2. Suppose $|z| > max\{|c|, 2\}$. Then $|Q_c^n(z)| > (1+\lambda)^n |z|$ and SO $|Q_c^n(z)| \to \infty,$ as n → ∞, where λ is a positive number.

Theorem 2. **(**General escape criterion**)** For general function, $Q_c(z) = z^n + c$, n = 1, 2, 3, …, where 0 < α ≤ 1 and c is in the complex plane. Define

$$z_1 = (1-\alpha)z + \alpha Q_c(z)$$

$$z_2 = (1-\alpha)z_1 + \alpha Q_c(z_1)$$

……..

$$z_n = (1-\alpha)z_{n-1} + \alpha Q_c(z_{n-1})$$

for n = 1, 2, 3,…. Thus, the general escape criterion is $\{|c|, (2/\alpha)^{1/n-1}\}$.

Corollary 3. Suppose that $|c| > (2/\alpha)^{1/n-1}$. Then the superior orbit SO(Q_c, 0, α_n) escape to infinity.

Corollary4. Suppose that for some k ≥ 0, we have $|z_k| >$ max $\{|c|, (2/\alpha)^{1/n-1}\}$. Then $|z_{k+1}| > (1+\lambda)|z_k|; so |z_n| \to \infty \, as \, n \to \infty$.

These corollary gives general algorithm for computing filled Superior Julia sets for the function of the form, $Q_c(z) = z^n + c$, n = 1, 2, 3, ….

Fig 1 and Fig 2 represents the superior Julia sets for quadratic and cubic maps using two superior iterative procedures.

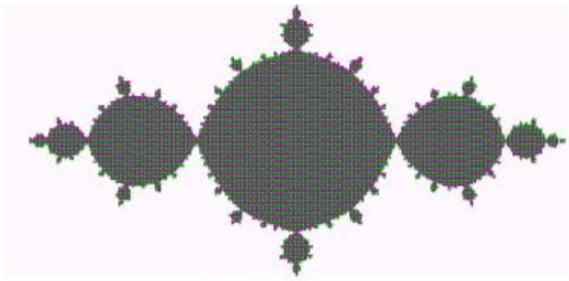

Figure 1: Superior Julia set for the Quadratic Map.

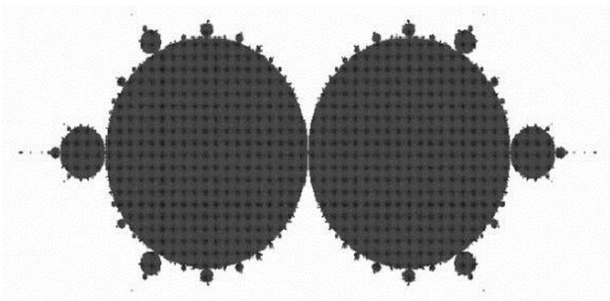

Figure 2: Superior Julia set for the Cubic Map.

In 2010, Chauhan et. al. [1] generated new Julia sets via Ishikawa iterates. The following theorem and their corollarygives the general I-superior escape criterion for the Julia sets.

Theorem 3 [4]. Let us assume that |z| > |c| > $2/s$; |z| > |c| > $2/s'$, where 0 < s < 1, 0 < s' < 1 and c is a complex number. Define

$$z_1 = (1-s)z + sQ_c(z)$$

$$z_2 = (1-s)z_1 + sQ_c(z_1)$$

…

…

…

$$z_n = (1-s)z_{n-1} + sQ_c(z_{n-1}),$$

where $Q_c(z)$ is a quadratic, cubic and biquadratic polynomial in term of s and n = 2, 3, 4,…

Then $|z_n| \to \infty \; as \; n \to \infty$.

Corollary 5. Suppose that |c| > $2/s$; |c| > $2/s'$. Then, the relative superior orbit of Ishikawa $RSO(Q_c, 0, s, s')$ escapes to infinity.

Corollary 6. Suppose that $|z| > \{|c|, 2/s, 2/s'\}$, then $|z_n| > (1+\lambda)^n |z|$ add $|z_n| \to \infty \; as \; n \to \infty$.

Fig 3 and Fig 4 represents I-Superior Julia sets for the quadratic and cubic maps.

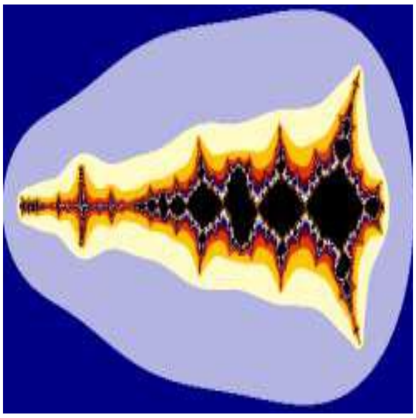

Fig. 3 Relative Superior Julia Set for s=0.4, s'=0.1, c=2.1+5.53i

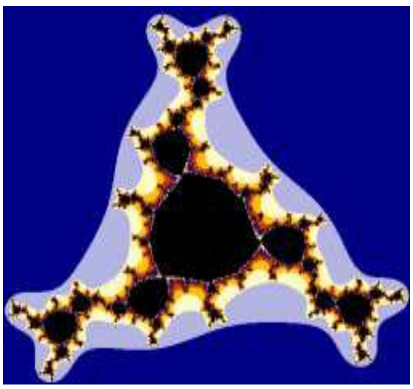

Fig. 4 Relative Superior Julia Set for s=1, s'=0.5, c= -0.146+1.54i

In 2014, Ashish et. al. [18] generates new Julia sets via Noor iterates. The following theorem gives the general escape criterion for the Julia sets and its corollaries further presents the escape criterion for computational purpose using Noor iterates.

Theorem 4 [13]. Suppose $|z| > |c| > 2/\alpha$, $|z| > |c| > 2/\beta$ and |z|>|c| >2/γ, where 0 <α< 1, 0 <β < 1 and 0 < γ < 1 and c is a complex number. Define

$$z_1 = (1-\alpha)z + \alpha Q_c(z)$$

$$z_2 = (1-\alpha)z_1 + \alpha Q_c(z_1)$$

…

…

$$z_n = (1-\alpha)z_{n-1} + \alpha Q_c(z_{n-1})$$

where $Q_c(z)$ can be a quadratic, cubic or biquadratic polynomial in terms of γ and n = 2, 3, . . ., then $|z_n| \to \infty \; as \; n \to \infty$.

Corollary 7. Suppose that |c| > 2/α, |c| > 2/β and |c| > 2/γ, then the orbit $NO(Q_c, 0, \alpha, \beta, \gamma)$ escapes to infinity.

Corollary 8. (Escape criterion). Let $|z| > \{|c|, 2/\alpha, 2/\beta, 2/\gamma\}$ then $|z_n| > (1+\lambda)^n |z|$ and $|z_n| \to \infty$ as n → ∞.

Corollary 9. Let for some k>0, we have $|z| > \{|c|, 2/\alpha, 2/\beta, 2/\gamma\}$. Then $|z_{k-1}| > (1+\lambda)^n |z_k|$ so that $|z_n| \to \infty \; as \; n \to \infty$.

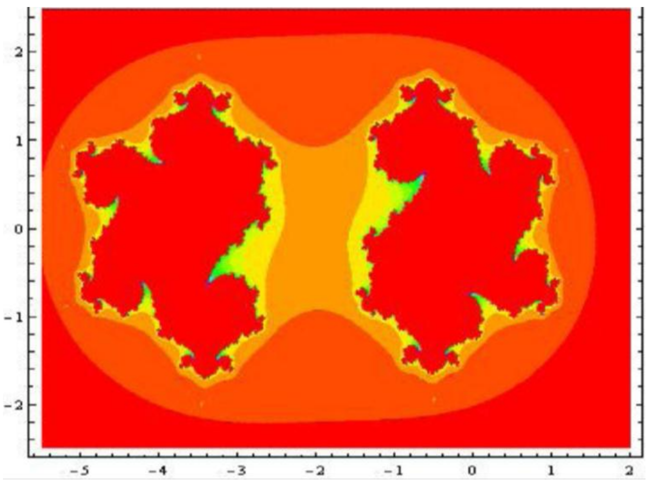

Fig. 5. Quadratic Julia set for, a = 0.2, b = c = 1, c = 0.05 + 0.05I.

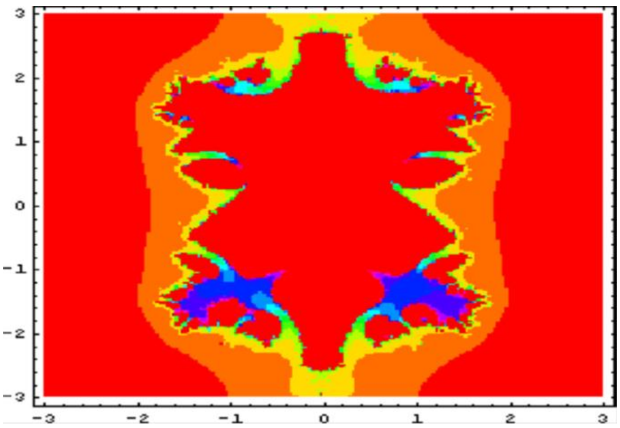

Fig.6. Cubic Julia set for, a = 0.1, b = 0.2, c = 0.3, c = -0.1–0.5I.

Recently, Kumari et. al. [19] generate new Superior Julia and Superior Mandelbrot sets for complex-valued polynomials such as quadratic, cubic and higher degree polynomials using SP orbit, which is an example of four-step iterative procedure.

The following theorem gives the general SP-iterates escape criterion for the Julia sets and its corollaries further presents the escape criterion for computational purpose using SP-iterates.

Theorem 5 [13]. For a general function $G_c(z)=z^n+c$, n =1, 2, 3…, where $0<\alpha<1$, $0<\beta<1$, $0<\gamma<1$, and c is a complex number. Define

$$z_1=(1-\alpha)u+\alpha G_c(u)$$

$$z_2=(1-\alpha)u_1+\alpha G_c(u_1)$$

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$z_n=(1-\alpha)u_{n-1}+\alpha G_c(u_{n-1})$, n = 1, 2, 3, 4, …

Then, the general Superior escape criterion is $\max\{|c|,(2/\alpha)^{1/n-1},(2/\beta)^{1/n-1},(2/\gamma)^{1/n-1}\}$.

Corollary 10. Suppose $|c|>(2/\alpha)^{1/n-1}$, $|c|>(2/\beta)^{1/n-1}$ and $|c|>(2/\gamma)^{1/n-1}$ exists. Then the orbit $SP(G_c,0,\alpha,\beta,\gamma)$ escapes to infinity.

Corollary 11. (Superior Escape Criterion). Let us assume that for some k ≥ 0, $|z_k|>\max\{|c|,(2/\alpha)^{1/k-1},(2/\beta)^{1/k-1},(2/\gamma)^{1/k-1}\}$, then $|z_k|>\lambda|z_{k-1}|$ and $|z_n|\to\infty$ as n→ ∞.

Fig 7 and Fig 8 represents the superior Julia sets for quadratic and cubic maps in SP-iterates.

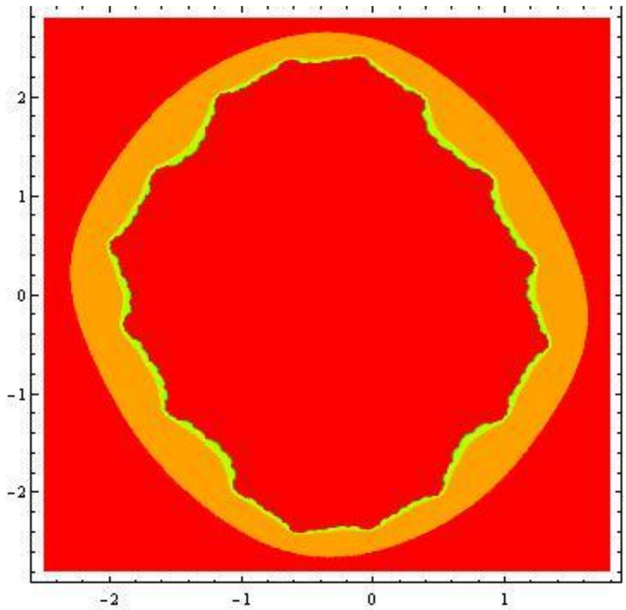


Fig. 7 Superior Julia set for c = -0.23+0.23I α = 0.3, β = 0.6, γ = 0.9

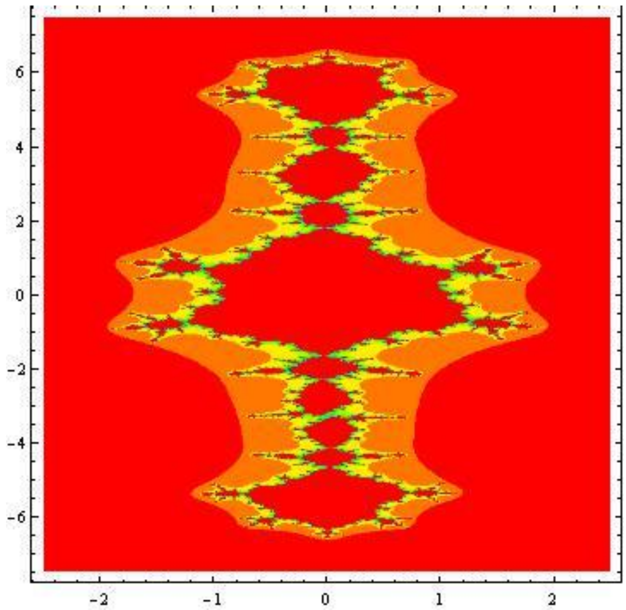


Fig. 8 Cubic Superior Julia set for c = - 0.2+0.6I α = 0.7, β = γ = 0.03

Mandelbrot Set [5]. The Mandelbrot set M is the collection of all parameters c for which the Filled Julia set of Q_c is connected, that is

$$M=\{c\in\mathbb{C}:K(Q_c)\,is\,connected\}.$$

The Superior Mandelbrot set SM for the quadratic map $Q_c(z)=z^2+c$ is defined as the collection of all

$c \in \mathbb{C}$ for which the filled Julia set is connected, that is

$$SM = \{c \in \mathbb{C} : \{Q_c^n(0)\}; n = 1, 2, 3, \ldots is connected\}$$

We choose the initial point 0, as 0 is the only critical point of Q_c [5].

The Mandelbrot set ISM for $Q_c(z) = z^n + c$, where n = 2, 3, 4,…, with respect to Ishikawa iterates is called I-superior Mandelbrot set[1].

The Mandelbrot set of $Q_c(z) = z^n + c$ where n = 2, 3, 4,…, with respect to Noor iterates is called Mandelbrot set [10].

Similarlly, the Mandelbrot set of $Q_c(z) = z^n + c$ where n = 2, 3, 4,…, with respect to SP-iterates is called Superior Mandelbrot set[13].

Escape criterion play a crucial role in the analysis and generation of Mandelbrot set, superior Mandelbrot set [6], I-superior Mandelbrot set [1], Mandelbrot set [10] and Superior Mandelbrot set [13]. The escape criterions studies in above section of Julia set are applicable in the generation of Mandelbrot set, superior Mandelbrot set, I-superior Mandelbrot set, Mandelbrot set and Superior Mandelbrot set.

Following are the general escape criterions of Mandelbrot set with some attractive figures:

► General escape criterion of Mandelbrot set for $Q_c(z) = z^n + c$, is $|z_k| > \max\{|c|, 2\}$. Fig. 9, shows the Mandelbrot sets generated by Picard Orbit.

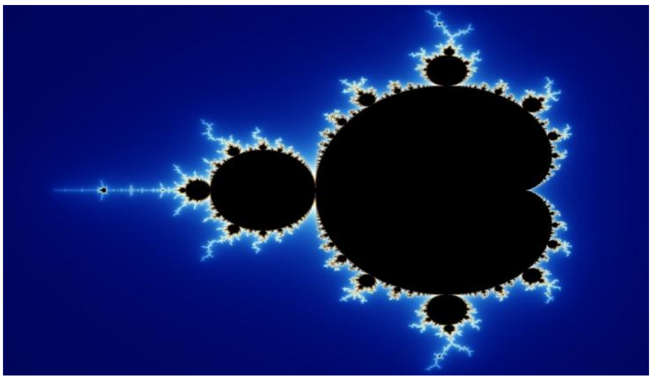


Figure 9: Mandelbrot set.

► General escape criterion of Mandelbrot set for $Q_c(z) = z^n + c$, is $|z_k| > \max\{|c|, (2/\alpha)^{1/k-1}\}$. Fig 10 and Fig 11, shows the Mandelbrot sets for quadratic and cubic maps generated by Maan iteration procedure.

Fig 10 Superior quadratic Mandelbrot set

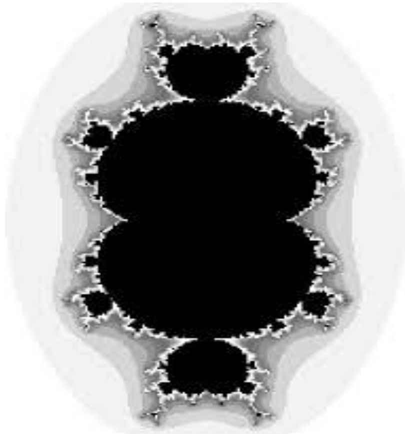


Fig 11: Superior cubic Mandelbrot set

► General escape criterion of I-superior Mandelbrot set for $Q_c(z) = z^n + c$, is $|z_k| > \max\{|c|, (2/\alpha)^{1/k-1}, (2/\beta)^{1/k-1}\}$. Fig 12 and Fig 13, shows the Mandelbrot sets for quadratic and cubic maps generated by three step iteration procedure.

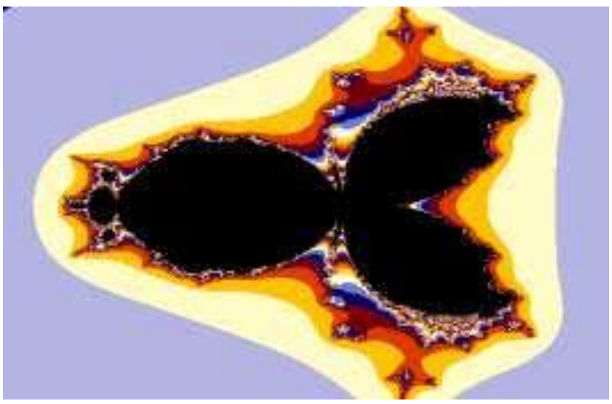


Fig 12 Mandelbrot set for quadratic map

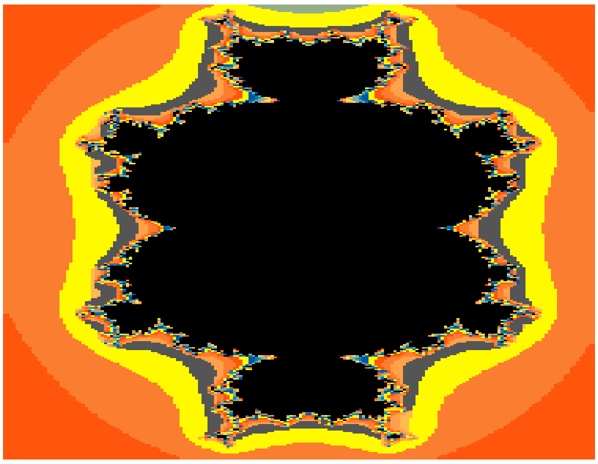


Fig 13 Mandelbrot set for cubic map

► General escape criterion of Noor Mandelbrot set for $Q_c(z)=z^n+c$, is $|z_k|>\max\{|c|,(2/\alpha)^{1/k-1},(2/\beta)^{1/k-1},(2/\gamma)^{1/k-1}\}$. Fig 14 and Fig 15, shows the Mandelbrot sets for quadratic and cubic maps generated by three step iteration procedure.

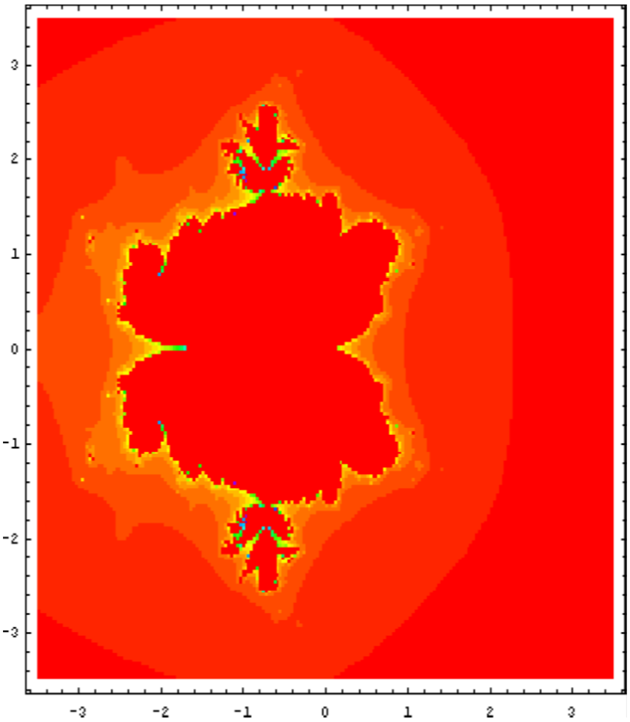


Fig. 14 Mandelbrot set (a=0.3,b=0.6, c=0.3)

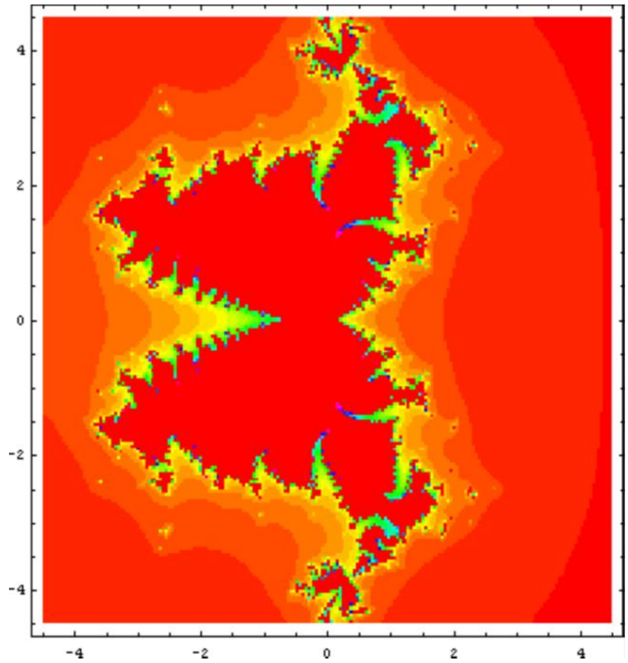


Fig. 15 Mandelbrot set (α=0.3, β=0.1, γ=0.3)

► General escape criterion of Mandelbrot set for $Q_c(z)=z^n+c$, is $|z_k|>\max\{|c|,(2/\alpha)^{1/k-1},(2/\beta)^{1/k-1},(2/\gamma)^{1/k-1}\}$. Fig 16 and Fig 17, shows the Mandelbrot sets for quadratic and cubic maps generated by SP iteration procedure.

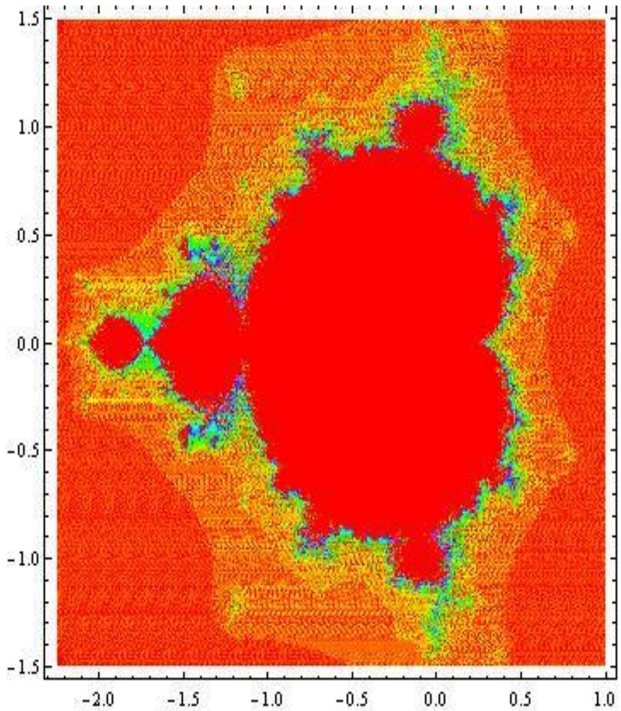


Fig. 16 Superior Mandelbrot set for n=3 (α =0.1, β = γ= 0.9)

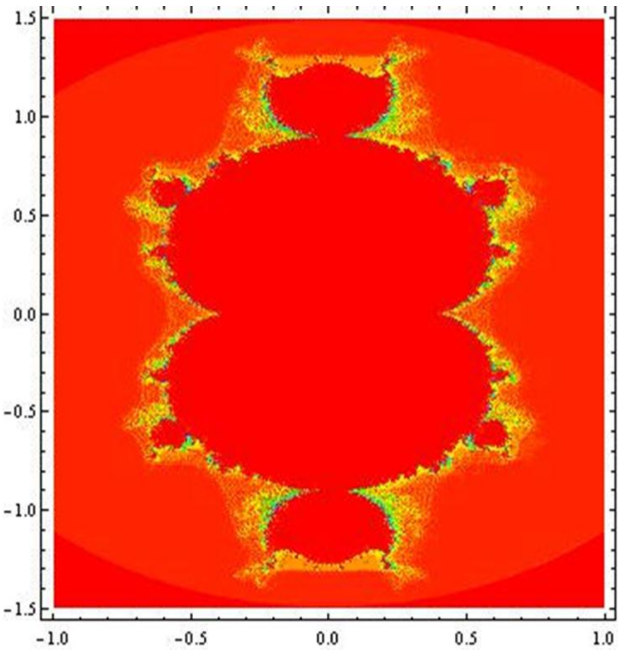


Fig. 17 Superior Mandelbrot set for n=3 (α = β = γ= 0.9)

The implimentation difference between the Julia set and the Mandelbrot set is the way in which the function is iterated. The Mandelbrot set iterates $z=z^2+c$ with z alwayes starting at 0 and varying the c value. The Julia set iterates $z=z^2+c$ for a fixed c values and varying z values. In other words, the Mandelbrot set is in the parameter space, or the c-plane, while the Julia set is in the dynamical space, or the z-plane. For a detailed study, one may refer to [15].

Contor Set

In mathematics, the Cantor set is a set of points lying on a single line segment that has a number of remarkable and deep properties. It was discovered in 1874 by Henry John Stephen Smithand introduced by German mathematician Georg Cantor in 1883 [5]. The Cantor set find a celebrating space in mathematical analysis ans its applicatins. The Contor set has many intersting properties and consequences in the filled of set theory, topology and fractal theory [6].

Recently, in 2010 Rani [14], introduction the superior Contor sets and presented them graphically by Devil's staircases. They generated new Contor sets by two methods. In one method, initiator is divided into three equal parts and either left segment or right segment of initiator is droped and in the other method, unequal division of initiator has been done.

The Cantor ternary set C is created by iteratively deleting the open middle third from a set of line segments. One starts by deleting the open middle third $(\frac{1}{3},\frac{2}{3})$ from the interval [0, 1], leaving two line segments: $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$ Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$. This process is continued ad infinitum, where the nth set is

$C_n=\frac{C_{n-1}}{3}\cup\left(\frac{2}{3}+\frac{C_{n-1}}{3}\right)$ for n ≥ 1, and $C_0=[0,1]$

The Cantor ternary set contains all points in the interval [0, 1] that are not deleted at any step in this infinite process:

$$C=\bigcap_{n=1}^{\infty}C_n\ .$$

The first six steps of this process are illustrated below.

This process of removing middle thirds is a simple example of a finite subdivision rule. The Cantor ternary set is an example of a fractal string.

Besides the sector mentioned above, chaos and fractals are the new frontiers of science and have come to play significant roles in the study of applicable areas of sciences, medicine, business, textile industries and several other areas of human activity [2].

Sierpinski Triangle (Gasket)

The Sierpinski triangle also called the Sierpinski gasket or the Sierpinski Sieve, is a fractal and attractive fixed set with the overall shape of an equilateral triangle, subdivided recursivelyinto smaller equilateral triangles. Originally constructed as a curve, this is one of the basic examples of self-similar sets, i.e., it is a mathematically generated pattern that can be reproducible at any magnification or reduction. It is named after the Polish mathematician Wacław Sierpiński [5, 11].

The Sierpinski triangle constructed from an equilateral triangle by repeated removal of triangular subsets:

1. Start with an equilateral triangle.

2. Subdivide it into four smaller congruent equilateral triangles and remove the central triangle.

3. Repeat step 2 with each of the remaining smaller triangles forever.

The first four steps of this process are illustrated below.

Each removed triangle (a trema) is topologically an open set. This process of recursively removing triangles is an example of a finite subdivision rule [5, 9].

Sierpinski Carpet

The Sierpinski carpet is a plane fractal first described by Wacław Sierpiński in 1916. The carpet is one generalization of the Cantor set to two dimensions. The same procedure is then applied recursively to the remaining 8 sub squares, ad infinitum. It can be realised as the set of points in the unit square whose coordinates written in base three do not both have a digit '1' in the same position[11].

The process of recursively removing squares is an example of a finite subdivision rule.

The Sierpinski carpet can also be created by iterating every pixel in a square and using the following algorithm to decide if the pixel is filled. The following implementation is valid C, C++, and most languages derived from C.

Koch Snowflake and Koch curve

The Koch snowflake also known as the Koch star, or Koch island is a mathematical curve and one of the earliest fractalcurves to have been described. It is based on the Koch curve, which appeared in a 1904 paper titled "On a continuous curve without tangents, constructible from elementary geometry" by the Swedish mathematician Helge von Koch [11].

The Koch snowflake can be constructed by starting with an equilateral triangle, then recursively altering each line segment as follows:

1. Divide the line segment into three segments of equal length.

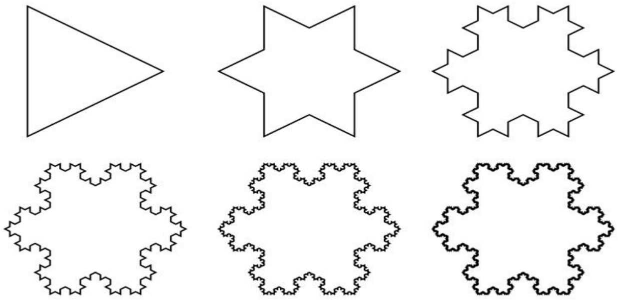
2. Draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.

3. Remove the line segment that is the base of the triangle from step 2.

After one iteration of this process, the resulting shape is the outline of a hexagram.

The Koch snowflake is the limit approached as the above steps are followed over and over again. The Koch curve originally described by Helge von Koch is constructed with only one of the three sides of the original triangle. In other words, three Koch curves make a Koch snowflake [5, 9].

The first six steps of this process are illustrated below.



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