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Fixed Point Theorem on Weakly Compatible Mappings in Metric Space

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Abstract – The point of this paper is to introduce a common fixed point theorem in a metric space which sums up the aftereffect of P.C. Lohani and V.H. Badshah utilizing the more vulnerable conditions, for example, at times weakly compatible mappings and related sequence instead of compatibility and completeness of the metric space. Additionally, the state of continuity of any of the mappings is being dropped.

Keywords: Fixed Point, Self-Maps, Compatible Mappings, Occasionally Weakly Compatible Mappings, Associated Sequence.

- X -

1. INTRODUCTION

Gerald Jungck gave a common fixed point theorem for driving mappings, which sums up the Banach's fixed point theorem. This outcome was additionally summed up and reached out in different manners by numerous creators. S. Sessa5 characterized weak commutativity and demonstrated common fixed point theorem for weakly commuting maps. Further G. Jungck1 presented the idea of good guides which is more vulnerable than weakly commuting maps. A short time later, Jungck and Rhoades4 characterized a more vulnerable class of guides known as weakly compatible maps. The idea of occasionally weakly compatible mappings in metric space is presented by A1-Thagafi and Shahzad10 which is generally broad among all the commutativity ideas.

The reason for this paper is to demonstrate a common fixed point theorem for four self-maps in which two sets are occasionally weakly compatible.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1. Let S and T be two self-mappings of a metric space (X, d) then S and T are said to be *commuting* on X if $STx = TSx$ for all x in X .

Definition 2.2. Two self-maps S and T of a metric space (X, d) are said to be *compatible mappings* if $\lim_{n\to\infty} d(STx_n, TSx_n)=0$, whenever is a sequence < x_n > in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, S and T are compatible mappings on X, then $d(STx, TSx) = 0$ when $d(Sx,Tx)$ for some x in X.

Definition 2.3. Two self-maps S and T of a metric space (X, d) are said to be *weakly compatible* if they commute at their coincidence point. i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

It is obvious that every compatible pair is weakly compatible but its converse need not be true.

Definition 2.4. Two self-maps S and T of a metric space (X , d) are said to be *occasionally weakly compatible* if S and T are commuting at some coincident points. That means S and T are not commuting at all coincidence points.

Weakly compatible mappings are occasionally weakly compatible mappings but converse is not true.

P. C. Lohani and V. H. Badshah proved the following theorem.

Theorem 2.5: Let P,Q, S and T be self-mappings from a complete metric space (X , d) into itself satisfying the following conditions

$$S(X) \subset Q(X) \text{ and } T(X) \subset P(X) \quad (2.5.1)$$

$$d(Sx,Ty) < \alpha \frac{d(Qy,Ty)[1+d(Px,Sx)]}{[1+d(Px,Qy)]} + \beta d(Px, Qy) \quad (2.5.2)$$

for all x, y in X.

one of P, Q, S and T is continuous (2.5.3)

the pairs (S, P) and (T, Q) are compatible on X (2.5.4)

then P, Q, S and T have a unique common fixed point in X.

Now we generalize Theorem2.5 using occasionally weakly compatible mappings and associated sequence.

Associated Sequence2.69: Suppose P, Q, S and T are self-maps of a metric space (X , d) satisfying the condition (2.5.1).Then for an arbitrary $x0 \in X$ such that $Sx0 = Qx1$ and for x_1, there exists a point x_2 in X such that Tx_1 and so on. Proceeding in the similar manner, we can define a sequence in X such that y_{2n} and $y_{2n+1} = Px_{2n+2} = Tx_{2n+1}$ for n. We shall call this sequence as an "Associated sequence of x_0, relative to the four self-maps P,Q,S and T.

Now we prove a lemma which plays an important role in our main Theorem.

Lemma2.7: Let P, Q, S and T be self-mappings from a complete metric space (X , d) into itself satisfying the conditions (2.5.1) and (2.5.2).Then the associated sequence{yn } relative to four self-maps is a Cauchy sequence in X.

Theorem 2.8 ([7]). Let A $\in$ $M_{m,m}(R_+)$, The followings are equivalent.

(i) A is convergent towards zero;

(ii) $A^n \to \bar{0}$ as n $\to \infty$;

(iii) the eigenvalues of A are in the open unit disc, that is $|\lambda| < 1$, for every $\lambda \in$ C with det(A - λ I) = 0;

(iv) the matrix I -A is nonsingular and $(I - A)^{-1} = I + A + \ldots\ldots. + A^n + \ldots\ldots$;

(v) The matrix I -A is nonsingular and $(I - A)^{-1}$ has nonnegative elements;

(v) for $A^n q \to \bar{0}$ and $qA^n \to \bar{0}$ as n $\to \infty$, for each q $\in R^m$

Remark: Some examples of matrix convergent to zero are

(a) any matrix A := $\begin{pmatrix} a & a \\ b & b \end{pmatrix}$ where a, b $\in R^+$ and a+b < 1;

(b) any matrix A := $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$ where a, b $\in R^+$ and a+b < 1;

(c) any matrix A := $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where a, b, c $\in R^+$ and max{a, c}<1;

For other examples and considerations on matrices which converge to zero, see [8] and [11].

Theorem 2.9 ([2]). Let (X, d) be a complete vector valued generalized metric space and the mapping f : X $\to$ X with the property that there exists a matrix A $\in M_{m,m}(R_+)$ such that

$d(f(x), f(y)) \leq Ad(x, y)$ for all x, y $\in$ X. If A is a matrix convergent towards zero, then

(1) Fix(f) = $\{x^*\}$;

(2) the sequence of successive approximations $\{x_n\}$ such that $x_n = f^n(x_0)$ is convergent and it has the limit x^*, for all $x_0 \in X$.

(3) One has the following estimation:

$$d(x_n, x^*) \leq A(I-A)^{-1} d(x_0, x_1)$$

3. MAIN RESULTS

Theorem 3.1: Let P, Q, S and T are self-maps of a metric space (X , d) satisfying the conditions

$S(X) \subset Q(X)$ and $T(X) \subset P(X)$ (3.1.1)

$d(Sx,Ty) < \alpha \frac{d(Qy,Ty)[1+d(Px,Sx)]}{[1+d(Px,Qy)]} + \beta d(Px, Qy)$ (3.1.2.)

for all x, y in X. where $\alpha, \beta \geq 0$, $\alpha + \beta < 1$

the pairs (S,P) and (Q,T) both are occasionally weakly compatible .

Further the associated sequence relative to four self-maps P, Q, S and T such that the sequence

Sx_0 ,Tx_1, Sx_2 ,Tx_3......Sx_{2n} ,Tx_{2n+1}.. converges to $z \in X$. Then P, Q, S and T have a unique common fixed point in X.

Proof: Using the condition (3.1.2), we have

Sx_{2n} tends to z, Qx_{2n+1} tends to z, Tx_{2n+1} tends to z.

Since (S,P)and (Q,T) both are occasionally weakly compatible. So there are points x, y in X such that

$Sx = Px$ and $Qy = Ty$

Using (3.1.2), we claim *Sx=Qy*

$$d(Sx,T) < \alpha \frac{d(Qy,Ty)[1+d(Px,Sx)}{[1+d\,(Px,Qy)} + \beta d(Px, Qy)$$

using the conditions *Sx* = *Px* and *Qy* = *Ty* , we get

$$d\,(Sx,Qy) \le \alpha \frac{d(Qy,Qy)[1+d(Sx,Sx)}{[1+d\,(Sx,Qy)} + \beta d(Sx, Qy)$$

$$d(Sx,Qy) \le \beta d(Sx,Qy)$$

$$\frac{d(Sx,Qy)}{d(Sx,Qy)} \le \beta, \text{ since } \beta \ge 0,$$

which is a contradiction.

Hence

Sx = Qy

Therefore

Sx = Qy = Px = Ty.

Suppose there is another point of coincidence say, w in X such that *Sz = Pz = w* then *Sz* = *Pz* = *Qy* = *Ty* , which gives *Sz* =*Sx* implies *z* = *x*

Hence *w* = *Sx* = *Px* for *w* ∊ *X* is the unique point of coincidence of P and S. By lemma (2.7), w is common fixed point of S and P .Hence *Sw* = *Pw* = *w*.

Similarly there exists a common fixed point of Q and T say *v* ∊ *X* such that *v* = *Qv* = *Tv.*

Suppose *w* ≠ *v* put *x* = *w* and *y* = *v* in the condition (3.1.2), we get

$$d(Sw,Tv) < \alpha \frac{d(Qv,Ty)[1+d(Pw,Sw)]}{[1+d\,(Pw,Qv)]} + \beta d(Pw, Qv)$$

$$d(w,v) < \alpha \frac{d(v,y)[1+d(w,w)]}{[1+d\,(w,v)]} + \beta d(w, v)$$

$$d(w,v) < \beta d(w, v)$$

This is a contradiction. Therefore *w v* .Hence *w* is a common fixed point of P, Q, S and T

Theorem (3.2):- Let $\in \Re$. Let C $\subset$ L_p be nonempty, ρ-closed and ρ-bounded. Let T: C $\to$ C be a ρ-contraction. Then, T has a unique fixed point f $\in$ C. Moreover, for any f $\in$ C., ρ (T^n (f) $-$ f) $\to$ 0 as n $\to \infty$, where T^n is the n^{th} iterate of T.

Proof: Since T is $\rho-$contraction, there exists $\alpha < 1$ such that

$$\rho\big(T(f) - T(g)\big) \le \alpha\, \rho\,(f-g), \text{ for all f, g} \in \text{C},$$

Let us fix $f_0 \in$ C. Since C is $\rho-$ bounded, hence

$$\delta_\rho\,(C) = \sup(\rho\,(f-g): f, g\, \in C) < \infty.$$

Observe that

$$\rho\big(T^{n+k}(f_0) - T^n(f_0)\big) \le \alpha\,\rho\,\big(T^{n+k-1}(f_0) - T^{n-1}(f_0)\big) \le \alpha^n\,\rho\,\big(T^k(f_0) - (f_0)\big)$$
$$\le \alpha^n\,\delta_\rho\,(C) \to 0,$$

for any n, k $\ge$1. Since α < 1 and δ_ρ (C)< ∞, we conclude that $\{T^n(f_0)\}$ is ρ-Cauchy. The ρ-completeness of $\mathfrak{A}$ implies the existence of $\bar{f} \in L^\rho$ Such that $\lim_{n\to\infty} \rho\,(T^n(f_0) - \bar{f}) = 0$ Because C is ρ-closed, we get $\bar{f} \in C$. Since

$$\rho\left(\frac{\bar{f}-T(\bar{f})}{2}\right) \le \rho\,(\bar{f} - T^n\,(f_0) + \rho\,(T^n\,(f_0) - T^n\,(\bar{f})$$
$$\le \rho\,(\bar{f} - T^n\,(f_0) + \alpha\rho\,(T^{n-1}\,(f_0) - \bar{f}\,) \to 0, \; as\, n \to \infty$$

Therefore, $T(\bar{f}) = \bar{f}$, which means that $\bar{f}$ is a fixed point of T. To prove the uniqueness part observe that if $T(f_1) = T(f_2)$, then

$$\rho(f_1 - f_2) = \rho(T(f_1) - T(f_2) \le \alpha\rho(f_1 - f_2) \qquad (3.2.1)$$

Since α < 1 and the right-hand side is finite, equality (3.3) can hold only if $f_1 = f_2$.

Theorem (3.3):- Let us assume that $\rho \in \Re$. Let C $\subset L_\rho$ be non empty, ρ-closed, and ρ-bounded. Let T: C $\to$ C be a point wise $\rho-$contraction or asymptotic point wise ρ-contraction. Then T has at most one fixed point $f_0 \in$ C. moreover, if f_0 is a fixed point of T, then the orbit $\{T^n$ (f)$\}$ ρ-converges to f_0 for any f $\in$ C.

Proof: Since every point wise ρ-contraction is an asymptotic point wise ρ-contraction, we can assume that T is an asymptotic point wise ρ-contraction, i.e., there exist a sequence of mappings α_n : C $\to$[0, ∞} such that

$\rho(T^n(f) - T^n(g)) \leq \alpha_n(f)\,\rho(f-g)$, for all f, g ∈ C,

where $\{\alpha_n\}$ converges pointwise to α : C → [0, 1). Let f_1, f_2 ∈ C be two fixed points of T. Then we have

$$\rho(f_1 - f_2) = \rho(T^n(f_1) - T^n(f_2)) \leq \alpha_n(f_1)\,\rho(f_1 - f_2)$$

for all n ≥ 1. If we let n → ∞, we will get

$$\rho(f_1 - f_2) \leq \alpha(f_1)\;\rho(f_1 - f_2)$$

Since $\alpha(f_1) < 1$ and C is ρ-bounded, we conclude that $\rho(f_1 - f_2) = 0$, i.e. $f_1 = f_2$. This proves that T has at most one fixed point. To prove the convergence, assume that f_0 is the fixed of T. Fix an arbitrary $f \in$ C. Let us prove that $\{T^n(f)\}$ ρ-converges to f_0. Indeed we have

$$\rho(T^{n+m}(f) - f_0) = \rho(T^{n+m}(f) - T^n(f_0)) \leq \alpha_n(f_0)\,\rho(T^n(f) - f_0)$$

for any n, m ≥ 1. Hence

$$\lim_{m\to\infty} sup\;\rho(T^{n+m}(f) - f_0) \leq \lim_{m\to\infty} sup\;\alpha_n(f_0)\,\rho(T^m(f) - f_0).$$

Since

$$\lim_{m\to\infty} sup\;\rho(T^{n+m}(f) - f_0) = \lim_{m\to\infty} \rho(T^m(f) - f_0), we\ get$$

$$\lim_{m\to\infty} sup\;\rho(T^m(f) - f_0) \leq \alpha_n(f_0) \lim_{m\to\infty} sup\;\rho(T^m(f) - f_0)$$

for any n ≥ 1. If we let n → ∞, we obtain

$$\lim_{m\to\infty} \rho(T^m(f) - f_0) \leq \alpha(f_0) \lim_{m\to\infty} sup\;\rho(T^m(f) - f_0).$$

Since $\alpha(x_0) < 1$, we get $\lim_{m\to\infty} sup\;\rho(T^n(f) - f_0) = 0$, which implies the desired conclusion $\lim_{m\to\infty} \rho(T^n(f) - f_0) = 0$ of $x0$ converges, the metric space (X , d) need not be complete.

CONCLUSION

Theorem3.1 is speculation of Theorem2.5 by prudence of the weaker conditions such as occasionally weakly compatibility of the sets (S,P) and (Q,T) instead of compatibility of the sets (S,P) and (Q,T). The continuity of any of the mappings is being dropped and the convergence of related sequence comparative with four self-maps S, P, Q and T is utilized set up the complete metric space.

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