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Weighted p-majorization and Stable vector

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Abstract - In this paper we generalized the notion of weighed p− majorization for any two vectors in R^n. We also define stable vector for a given set of vectors. Then we prove existence of stable vector for n = 2. And finally we give some conditions on a set of vectors for existence of stable vector for n ≥ 3. we also give some examples for illustration our theorems.

Keywords - Majorization, Lorenz Curve, Non-Vanishing

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INTRODUCTION

The notion of majorization has been introduced while studying the topic such as wealth distribution, inequalities etc. In the early part of the twentieth century, Lorenz introduced a curve, that describes the wealth (or income) distribution of a population. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be denote the income profile of a population of size *n*. Then the line joining origin and the points $\left(\frac{k}{n}, \frac{S_k}{S_n}\right), S_k = \sum_{j=1}^{k} \alpha_j^{\downarrow}, 1 \le k \le n$, where $\alpha_1^{\downarrow}, \alpha_2^{\downarrow}, \cdots, \alpha_n^{\downarrow}$ be the decreasing rearrangement of the components of α, is called Lorenz curve of α. If total income or wealth of the population is uniformly distributed among the population then the Lorenz curve is a straight line, otherwise, the curve is convex. Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ be denote the income of a population of size n, in day 1 and day 2 respectively. If *x* is majorized by *y*, then Lorenz curve of *x* is closer to the straight line of uniform distribution than the Lorenz curve of *y*. This tech- nique is used in many different [6, 7] such as describing inequality among the size of individuals in ecology, in studies of biodiversity, business modeling, etc. The Lorenz curve also provides a different tool for estimating the distributional dimensions of energy consumption.

Many authors generalized and studied the majorization by introducing different parameters [1, 2, 5]. In 1947, by introducing weighted *p*-majorization for a pair of vectors that are in a similar order (either increasing or decreasing) Fuchs [3], proved an equivalent condition for weighted *p* majorization using continuous convex functions. In 1997, Pˇecar´c and Abramovich [4] discussed an analogs of Fuchs result in which order of one of vector in the pair can be relaxed. In general the wealth (or income) profiles of individuals in a population may not be in a similar order. This motivates us to define majorization for a pair of vectors in R^n by introducing a weight function with out having any order restriction. We use technique of Lorenz as a tool to introduce stability for a given set of points by assigning a proper weight. We prove the existence of a stable vector for a given set of vectors in R^n with respect to a given weight. Finally, we give examples to illustrate our techniques.

2. WEIGHTED P-MAJORIZATION

Let R_n^+ be the positive cone of R^n, the set of all vectors of R^n with positive coordinates. Let $p = (p_1, p_2, \cdots, p_n) \in R_n^+$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a vector in R^n with $\sum_{i=1}^{n} p_i \alpha_i \neq 0$. Define $\alpha^p := (\alpha_1^p, \alpha_2^p, \dots, \alpha_n^p)$ where $\alpha_i^p = \frac{p_i \alpha_i}{\sum_{i=1}^{n} p_i \alpha_i}$ $(1 \le i \le n)$. We call such a vector α as non-vanishing p-vector of R^n.

Definition 2.1. *Let $p = (p_1, p_2, \dots, p_n)$ be in* R*n and $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ be two non-vanishing p-vectors of* R*n. We say that x is p-weighted majorized by y if x^p is majorized by y^p. We denoted it by by $x \prec_p y$.*

Let $p \in$ R*n* and *x, y* be two non-vanishing *p*-vectors in R*n*. If $x \prec_p y$, then from Figure [1] it is to be noted that Lorenz curve of x^p is closer to the Lorenz curve of uniform distribution than the Lorenz curve of y^p. In the other words, we say that the vector x^p is stable than than the vector y^p.

Definition 2.2. *Let p be a fixed vector in* R*n and let $S = \{x_1, x_2, x_3, \dots, x_m\}$ denotes a set of any m vectors in* R*n with non-negative coordinates. A* point x_j in the set S is said to be a p-stable vector of the set S if $x_j \prec_p x_i$ for all $i \in \{1, 2, . . . , m\}$.

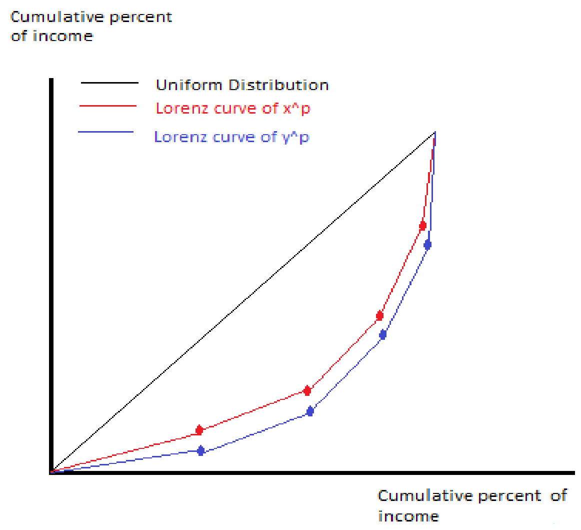

Figure 1 : Lorenz curve

Theorem 2.3. *Let p be a fixed vector in R^2 and let S = $\{x_1, x_2, x_3, . . . , x_m\}$ be a set of any m vectors in R^2 with positive coordinates. Then there exists a stable vector of the set X with respect to weight p i.e. there exist a vector x_io ($1 \le i_o \le m$) such that $x_io \prec_p x_i$ for all $i \in \{1, 2, . . . , m\}$.*

Proof. Let $p = (p_1, p_2)$ and $x_i = (x_{i1}, x_{i2})$ for $i \in \{1, 2, . . . , m\}$. Then

$$x_i^p = (x_{i1}, x_{i2}) = \left(\frac{p_1 x_{i1}}{\sum_{j=1}^{2} p_j x_{ij}}, \frac{p_2 x_{i2}}{\sum_{j=1}^{2} p_j x_{ij}}\right) \quad (1 \le i \le n)$$

Further, for each i $(1 \le i \le n)$, there exist a permutation σ^i on $\{1,2\}$ such that

$$x_i^{p\downarrow} = (x_{i1}^{p\downarrow}, x_{i2}^{p\downarrow}) = \left(\frac{p_{\sigma^i(1)} x_{i\sigma^i(1)}}{\sum_{j=1}^{2} p_j x_j^i}, \frac{p_{\sigma^i(2)} x_{i\sigma^i(2)}}{\sum_{j=1}^{2} p_j x_j^i}\right)$$

Consider the set $\phi = \{\frac{p_{\sigma^i(1)} x_{i\sigma^i(1)}}{\sum_{j=1}^{2} p_j x_j^i} : 1 \le i \le m\}$. As the set ϕ has a minimum, let it be at i_o where $i_o \in \{1,2,\cdots,n\}$. Thus we get $x_{i01}^{p\downarrow} \le x_{i1}^{p\downarrow}$ for all $i \in \{1,2,\cdots,n\}$. Therefore $x_io \prec_p$ x_i *for all* $i \in \{1, 2, . . . , n\}$. *Hence* x_{i_0} *is p-stable vector of the set S.*

The following example shows that for $n \ge 3$ it is not always possible to find a *p*-stable vector of a given set of vectors in R*n* with positive coordinates and a given vector *p* in R_+^n.

Example 2.4. Take $n = 3, m = 2, p = (1,1,1), x_1 = \{3,1,1\}$ and $x_2 = \{2.7,1.5,0.8\}$. Then $x_1^p = \left\{\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right\}$ and $x_2^p = \left\{\frac{2.7}{5}, \frac{1.5}{5}, \frac{0.8}{5}\right\}$. By a direct calculation, one can show that neither x_1^p majorized by x_2^p nor x_2^p majorized by x_1^p. Thus neither x_1 or x_2 is a p-stable vector of the set containing the vectors x_1 and x_2 with respect to the weight p.

Example 2.5. Take $n = 3, m = 2, p = (1,2,3), x_1 = \{3,4,5\}$ and $x_2 = \left\{\frac{2,5,14}{3}\right\}$. Then $x_1^p = \left\{\frac{3}{26}, \frac{8}{26}, \frac{15}{26}\right\}$ and $x_2^p = \left\{\frac{2}{26}, \frac{10}{26}, \frac{14}{26}\right\}$. It is easy to verify that neither neither x_1^p majorized by x_2^p nor x_2^p majorized by x_1^p. Thus neither x_1 or x_2 is a p-stable vector of the set containing the vectors x_1 and x_2 with respect to the weight p.

Let $p(p_1, p_2, \cdots, p_n)$ be a fixed vector in R_+^n and $S = \{x_1, x_2, x_3, . . . , x_m\}$ be a set if vectors in R^n. Suppose Suppose $x_i = (x_{i1}, x_{i2}, . . . x_{in})$ for $i = 1, 2, . . . , m$. Then

$$x_i^p = (x_{i1}^p, x_{i2}^p, \cdots, x_{in}^p) = \left(\frac{p_1 x_{i1}}{\sum_{j=1}^{n} p_j x_j^i}, \frac{p_2 x_{i2}}{\sum_{j=1}^{n} p_j x_j^i}, \cdots, \frac{p_n x_{in}}{\sum_{j=1}^{n} p_j x_j^i}\right), i = 1,2,\cdots,m.$$

For each i, there exist a permutation σ^i on $\{1,2,\cdots,n\}$ such that

$$x_i^{p\downarrow} = (x_{i1}^{p\downarrow}, x_{i2}^{p\downarrow}, \cdots, x_{in}^{p\downarrow}) = \left(\frac{p_{\sigma^i(1)} x_{i\sigma^i(1)}}{\sum_{j=1}^{n} p_j x_j^i}, \frac{p_{\sigma^i(2)} x_{i\sigma^i(2)}}{\sum_{j=1}^{n} p_j x_j^i}, \cdots, \frac{p_{\sigma^i(n)} x_{i\sigma^i(n)}}{\sum_{j=1}^{n} p_j x_j^i}\right)$$

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Consider the set $\phi = \{\frac{p_{\sigma^i(1)} x_{i\sigma^i(1)}}{\sum_{j=1}^{2} p_j x_j^i} : 1 \le i \le m\}$. This set ϕ has a minimum say at i_o where $i_o \in \{1,2,\cdots,n\}$. Thus we get $x_{i01}^{p\downarrow} \le x_{i1}^{p\downarrow}$ for all $i \in \{1,2,\cdots,m\}$. *Conditions on vectors….*

(1) $x_{i02}^{p\downarrow} \le x_{i2}^{p\downarrow} + \Delta_1^n$ for $i = 1,2,\cdots,m$, where $\Delta_1^i = x_{i1}^{p\downarrow} - x_{i01}^{p\downarrow}$.

(2) $x_{i03}^{p\downarrow} \le x_{i3}^{p\downarrow} + \Delta_2^n$ for $i = 1,2,\cdots,m$, where $\Delta_2^i = \sum_{j=1}^{2} x_{ij}^{p\downarrow} - \sum_{j=1}^{2} x_{i0j}^{p\downarrow}$.

(3) $x_{i04}^{p\downarrow} \le x_{i4}^{p\downarrow} + \Delta_3^n$ for $i = 1,2,\cdots,m$, where $\Delta_2^i = \sum_{j=1}^{3} x_{ij}^{p\downarrow} - \sum_{j=1}^{3} x_{i0j}^{p\downarrow}$.

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$(n-2)x^{p\downarrow}_{i0(n-1)} \leq x^{p\downarrow}_{i(n-1)} + \Delta^{n}_{n-2}$ for $i = 1,2,\cdots,m,$

where $\Delta^{i}_{n-2} = \sum_{j=1}^{n-2} x^{p\downarrow}_{ij} - \sum_{j=1}^{n-2} x^{p\downarrow}_{i0j}$.

If the vectors satisfy these conditions then we get a stable vector say x_{i_0} with respect to p.

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