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A Research on Some Application and Solutions of Fixed Point Theory

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Abstract – Fixed point theory is a captivating subject, with a tremendous number of utilizations in different fields of arithmetic. Possibly because of this transversal character, I have constantly encountered a few troubles to discover a book (except if explicitly committed to fixed points) treating the contention in a unitary manner.

Fixed point theory worries about an exceptionally straightforward, and fundamental, numerical setting. For a function f that has a set X as area and range, a fixed point off is a point x of X for which f(x) = x. Banach's and of Brouwer's theorems are the two major theorems concerning fixed points. In Banach's theorem, X L' a .jmplete metric space with metric d and $f: X \to X$ is required to be a constriction, that is, there must exist $\alpha < 1$ to such an extent that $d(f(x), f(y)) = \alpha d(x,y)$ for each of the $x, y \in X$. The end is that f has a fixed point, in certainty precisely one of them. Brouwer's theorem expects X to be the shut unit ball in an Euclidean space and $f: X \to X$ to be a consistent function. Again we can infer that f has a fixed point. Be that as it may, for this situation the set of fixed points need not be a solitary point, in truth each shut nonempty subset of the unit ball is the fixed point set for some map. The theorems of Banach and Brouwer delineate the distinction between the two important parts of fixed point theory: metric fixed point theory and topological fixed point theory.

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INTRODUCTION

The nearness or nonappearance of fixed point is a natural property of a function. Anyway numerous important and additionally adequate conditions for the presence of such points include a blend of algebraic request theoretic or topological properties of mapping or its space. Fixed point theory frets about an extremely straightforward and essential scientific setting.

A point is frequently called fixed point when it stays invariant, regardless of the sort of change it experiences. For a function f that has a set X as both space and range, a fixed point is a point $x \in X$ for which f(x) = x. Two central theorems concerning fixed points are those of Banach and of Brouwer.

We start our examinations with the accompanying known definitions;

DEFINITION 1 ; Let X be a non-void set A mapping $d: X \times X \to R$ (the set of reals) is said to be a metric (or separation function) iff d fulfills the accompanying axioms:

(M-1) $d(x,y) \geq 0$ for each of the , $x, y \in X$

(M-2) d(x, y) =0 iff x = y,

(M-3) d(x,y) = d(y,x) for each of the $x, y \in X$,

(M-4) $d(x,y) \leq d(x,z) + d(z,y)$ for each of the $x, y, z \in X$.

In the event that d is metric for X, at that point the arranged pair (X,d) is known as a metric space and d(x,y) is known as the separation among x and y.

DEFINITION 2 : Let (X,d) be a metric space and let A be a non-void subset of X. At that point the distance across of A, meant by $\delta(A)$, is defined by; $\delta(A) = \sup\{d(x,y) : x, y \in A\}$ that is y the breadth of An is the supremum of the set of all distances between points of A.

DEFINITION 3 : The separation between a point $p \in X$ and a subset An of metric space X is meant and defined by $d(p, A) = \inf\{d(p,x) : x \in A\}$. It is apparent that d(p, A) = 0 if $p \in A$.

DEFINITION 4 : The separation between two non-void subsets An and B of a metric space X is signified and defined as; $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

DEFINITION 5 : Let (X,d) be a metric space and A be any subset of X. A point $x \in X$ is an inside point of An if there exists r > 0, to such an extent that $x \in S_r(x) \subset A$.

DEFINITION 6 : Let (X,d) be a metric space and A be any subset of X. A point $x \in X$ is an outside point of An if there exists an open circle Sr(x), with the end goal that $S_r(x) \subset A^c$, or $S_r(x) \bigcap A = \phi$.

DEFINITION 7 : Let A be a nonempty subset of a metric space (X,d). A point $x \in X$ is said to be the limit point of An if x is neither an inside point of A nor an outside point of A. The limit of A will be indicated by ∂A.

DEFINITION 8 : A grouping of components $x_1, x_2, x_3 \ldots\ldots x_n \cdots$ in a metric space X is said to merge to a component $x \in X$ if the sequence of real numbers converges to zero as $n \to \infty$, for example $\lim_{n\to\infty} x_n = x$.

DEFINITION 9 : Let (X,d) be a metric space and let {pn} be a grouping of points in X, at that point it is said to be a Cauchy arrangement in X if and if for each $\epsilon > 0$ there exists a positive number $n(\epsilon)$ to such an extent that, $m, n \geq n(\epsilon) \Rightarrow d(p_m, p_n) < \epsilon$.

Obviously every united arrangement in a metric space is a Cauchy grouping yet the opposite need not be valid. A metric space (X, d) is said to be complete if and just if each Cauchy succession in X meets to a point in X.

DEFINITION 10 ; A self mapping T of a metric space (X, d) is said to be Lipschitzian if for every one of the $x, y \in X$ and $\alpha \geq 0$

$$d(f(x), f(y)) \leq \alpha\ d(x, y). \quad (1)$$

T is said to be constriction on an if $\alpha \in [0, 1)$ and nonexpansive if $\alpha = 1$. A contraction mapping is constantly nonstop.

In 1922, S. Banach's withdrawal principle showed up and this was known for its basic and exquisite confirmation by utilizing the Picard's cycle in a complete metric space. Banach's fixed point theorem states;

THEOREM 1: Let X be a complete metric space with metric d and $f : X \to X$ is required to be a contractionf that is there must exists $\alpha < 1$ to such an extent that,

$$d(f(x), f(y)) \leq \alpha\ d(x, y),\ \forall\ x, y\ \in\ X, \quad (2)$$

the end is that, f has a fixed point, in actuality precisely one.

Verification: Let $x \in X$ be an arbitrary element. Starting from x we structure the cycles,

$$x_1 = fx,\ x_2 = fx_1,\ x_3 = fx_2 \ldots\ldots,\ x_n = fx_{n-1} \ldots$$

We check that {xn} is a Cauchy succession. We have,

$$d(x_1, x_2) = d(fx, fx_1) \leq \alpha\ d(x, x_1) = \alpha\ d(x, fx)$$

$$d(x_2, x_3) = d(fx_1, fx_2) \leq \alpha\ d(x_1, x_2) \leq \alpha^2\ d(x, fx)$$

$$d(x_3, x_4) = d(fx_2, fx_3) \leq \alpha\ d(x_2, x_3) \leq \alpha^3\ d(x, fx).$$

When all is said in done, for any positive number n, $d(x_n, x_{n+1}) \leq \alpha^n\ d(x, fx)$. Also, for any positive whole number p,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \alpha^n\ d(x, fx) +\ \alpha^{n+1} d(x, fx) + \ldots\cdots +\ \alpha^{n+p-1}\ d(x, fx) \\ &= (\alpha^n\ +\ \alpha^{n+1} \ldots\cdots +\ \alpha^{n+p-1})\ d(x, fx) \\ &= \frac{\alpha^n - \alpha^{n+p}}{1-\alpha}\ d(x, fx) < \frac{\alpha^n}{1-\alpha}\ d(x, fx),\ \ (\text{since } 0 < \alpha < 1). \end{aligned}$$

Since $\alpha < 1$, the above relation demonstrates that $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$. Therefore {xn} is a Cauchy arrangement. Since X is complete, the grouping {xn} meet to a point X0 (state) in X. Presently we demonstrate that fx0 = x0, for this by triangle disparity we have,

$$\begin{aligned} d(x_0, fx_0) &\leq d(x_0, x_n) + d(x_n, fx_0) \\ &= d(x_0, x_n) + d(fx_{n-1}, fx_0) \\ &\leq d(x_0, x_n) +\ \alpha\ d(x_{n-1}, x_0) \to 0 \text{ as } \ n \to \infty. \end{aligned}$$

So fxo = x0. Consequently Xo is a fixed point off. Uniqueness can be effectively check utilizing logical inconsistency technique. So f has a unique fixed point in X.

FIXED POINT THEOREMS IN METRIC SPACES

The term metric fixed point theory alludes to those fixed point theoretic results in which geometric conditions on the basic spaces or potentially mappings assume an essential job. For as far back as a quarter century metric fixed point theory has been a thriving zone for some mathematicians. Despite the fact that a generous number of complete results currently have been found, a couple of inquiries lying at the core of the theory stay open and there are numerous unanswered inquiries with respect as far as possible to which the theory might be broadened.

It is outstanding since the paper of Kannan (1968) that there exists maps that have an irregularity in their area yet have fixed points. Be that as it may, for each situation the maps included were constant at the fixed point. In 1998, Pant presented the idea of complementary nonstop maps and utilizing the thought of corresponding progression he improved numerous results. The present paper is gone for getting a typical fixed point theorem, in which fixed point might be a point of intermittence. Further, we give an example which shows our attestation.

Before demonstrating our fundamental result we review here some outstanding definitions;

DEFINITION 1: Two self maps An and B of a metric space [Xyd) are said to be weak'compatible in the event that they drive at their occurrence points, for example Hatchet = Bx infers ABx = BAx.

DEFINITION 2: Two self maps A and B of a metric space (X,d) are said to be compatible if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever {x_n} is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, for some $x \in X$.

DEFINITION 3: Two self maps A and B of a metric space (X,d) are said to be compatible maps of type (A) if $\lim_{n\to\infty} d(ABx_n, BBx_n) = 0$ & $\lim_{n\to\infty} d(BAx_n, AAx_n) = 0$ whenever {x_n} is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, for some $x \in X$.

DEFINITION 4 *: Two self maps A and B of a metric space (X,d) are said to be compatible maps of type (B) if* $\lim_{n\to\infty} d(ABx_n, BBx_n) = 0$ *whenever* {x_n} *is a sequence such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, *for some* $x \in X$.

DEFINITION 5: *Two self-maps A and B of a metric space (X,d) are said to be compatible maps of type (C) if* $\lim_{n\to\infty} d(BAx_n, AAx_n) = 0$ *whenever* {x_n} *is a sequence such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, *for some* $x \in X$.

DEFINITION 6: *Two self-maps A and B of a metric space (X,d) are said to be compatible maps of type (P) if* $\lim_{n\to\infty} d(AAx_n, BBx_n) = 0$ *whenever* {x_n} *is a sequence such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, *for some* $x \in X$.

Definition 7: *A pair {A,S) of self maps of a metric space (X,d) is said to be reciprocal continuous if* $\lim_{n\to\infty} ASx_n = Ax$ *and* $\lim_{n\to\infty} SAx_n = Sx$, *whenever there exists a sequence* {x_n} *in X such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$, *for some* $x \in X$.

If the maps A and B are continuous then they are obviously reciprocal continuous but the converse need not be true as shown by Pant [95].

BANACH'S FIXED POINT THEOREM

First we take a gander at the issue to locate a fixed point for a genuine esteemed persistent function $f : \mathbb{R} \to \mathbb{R}$ in the spirit of Banach's fixed point theorem. We at that point need f to be a constriction implying that there is a positive genuine number c under 1 with the end goal that for any pair x, y of points the separation between the pictures under f of these points are nearer by a division c than the separation between the points x and y. In recipes this implies

$$|f(x) - f(y)| \leq c|x - y|$$

for self-assertive $x, y \in \mathbb{R}$. The end from Banach fixed point theorem is that there is a unique fixed point for f. This can be found by simply fixing any component $z \in \mathbb{R}$ and afterward shaping the sequence1 $(T^n(z))_{n=1}^{\infty}$. This is a merging grouping with the fixed point as the breaking point. Then again there is no confinement on the space off being a raised reduced set.

We first, state and demonstrate some broad perceptions.

Theorem 1.1. Give *T* a chance to be a nonstop mapping on a Banach space X. At that point the accompanying explanations remain constant:

1- *If there exist* $x, y \in X$ *such that*

$$\lim_{n\to\infty} T^n(x) = y$$

then y is a fixed point for T, i.e. T(y) = y.

2- *If T(X) is a compact set in X and for each* $\epsilon > 0$ *there exists* $a\ x_\epsilon \in X$ *such that*

$$\|T(x_\epsilon) - x_\epsilon\| < \epsilon$$

then T has a fixed point.

Proof. Set $y_n = T^n(x),\ n = 1, 2, \ldots$. If *T* is a continuous mapping then

$$T(y) = T(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} y_{n+1} = y,$$

which proves the first statement.

Assume that the assumptions are satisfied. Then for $n = 1, 2, \ldots$ there are $x_n \in X$ such that

$$\|T(x_n) - x_n\| < \frac{1}{n}. \quad (1)$$

T(X) is a compact set implies that there exits a convergent subsequence $(T(x_{n_k}))_{k=1}^{\infty}$ of $(T(x_n))_{n=1}^{\infty}$. Call the limit point *x*. Then *x* is a fixed point for *T* since also the sequence $(x_{n_k})_{k=1}^{\infty}$ converges to *x* according to (1) and *T* is continuous.

BROUWER AND SCHAUDER FIXED POINT THEOREMS

We begin by detailing Brouwer fixed point theorem.

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Theorem 1.4 (Brouwer's fixed point theorem). Expect that K is a smaller curved subset of $\mathbb{R}^n$ and that $T : K \to K$ is a ceaseless mapping. At that point T has a fixed point in K.

Note that it doesn't pursue from Brouwer fixed point theorem that the fixed point is unique. Consider for example the personality administrator on a minimized raised set K in Rn for which each $x \in K$ is a fixed point.

Example 1: Take a road map for Goteborg and spot it on the floor of the address room at Chalmers. At that point there will be a point on the map that agrees with the relating point in Goteborg. This pursues from both Banach's fixed point theorem and Brouwer's fixed point theorem, where the previous theorem additionally gives that the point is unique. Demonstrate this to yourself!

Example 2: Let T_α mean the turn a degrees around the middle for a shut plate K of range 1. At that point Brouwer's fixed point theorem gives the presence of a fixed point for T_α (obviously it is needless excess to utilize a fixed point theorem to see that) while Banach's fixed point theorem can't be connected directly2 since T_α isn't a withdrawal. Clearly the middle is a fixed point yet Brouwer's fixed point theorem likewise reveals to us that it is beyond the realm of imagination to expect to make the pivot with a ceaseless disfigurement of the circle into itself so that the made mapping has no fixed point.

We note that

- (speculation of Brouwer's fixed point theorem): If there exists a homeomor-phism, for example a consistent bijection with nonstop reverse, between a minimal raised set K in $\mathbb{R}^n$ and a set $\tilde{K}$, call the homeomorphism φ, and $\tilde{T} : \tilde{K} \to \tilde{K}$ is a constant mapping at that point $\tilde{T}$ has a fixed point. To see this think about the mapping $T = \varphi^{-1} \circ \tilde{T} \circ \varphi$.

Exercise: Prove that $\tilde{T}$ has a fixed point.

- it is sufficient to demonstrate Brouwer fixed point theorem for the situation $K = \overline{B(0,1)}$.

There are numerous evidences for Brouwer's fixed point theorem, both logical and topological. We simply sketch one proof. Accept that $K = \overline{B(0,1)}$ and that T has no fixed point. Define the mapping $A : \overline{B(0,1)} \to \overline{B(0,1)}$ as pursues: For each inward point x in $\overline{B(0,1)}$ let x denote the point on the limit $\partial B(0,1)$ that is the intersection of the beam from T(x) through x and the limit $\partial B(0,1)$. The beam is in every case well-defined since T has no fixed point. Presently set

$$A(x) = \begin{cases} \tilde{x} & \text{if } x \in B(0,1) \\ x & \text{if } x \in \partial B(0,1) \end{cases}$$

At that point A will be a consistent mapping from $\overline{B(0,1)}$ into $\partial B(0,1)$ (check this!) such that $A|_{\partial B(0,1)} = I|_{\partial B(0,1)}$. The problem to demonstrate that T has no fixed point is presently reformulated as to demonstrate that there is no nonstop mapping $A : \overline{B(0,1)} \to \partial B(0,1)$ with the end goal that $A|_{\partial B(0,1)} = I|_{\partial B(0,1)}$. The statement that there is no such mapping is profound yet never the less instinctively obvious4. Consider, for n = 2, the case with a versatile film fixed on a round edge. The presence of a mapping A suggests that it should be conceivable to misshape the film ceaselessly so that it ought to harmonize with the edge without being cracked. For fixed $x \in B(0,1)$ the mapping $t \mapsto (1-t)x + tA(x),\ t \in [0,1]$ portrays how this point on the film is moved from x at t = 0 to $A(x) \in \partial B(0,1)$ at t = 1, under the distortion. Remember that the film ought to be fixed at the edge!!!

We present Perron's theorem as an utilization of Brouwer's fixed point theorem. Schauder's fixed point theorem will be connected with regards to nonlinear differential/indispensable conditions to demonstrate the presence of arrangements.

REFERENCES

1. Aamri, M. and Moutawakil, D. E.l. (2002). *Some new common fixed point theorems under strict contractive conditions,* J. Math. Anal. Appl., 207, pp. 181-188.

2. Bouhadjera, H. (2005). *General common fixed point theorem for compatible mappings of type (C),* Sarajevo J. Math., 1 (2), pp. 261-270.

3. Brouwer, L. E. J. (1910). *Over ceen enduing, continue transformation van oppervlaleben in zide self,* Amsterdam Abad. Verl., 17, pp. 741-752.

4. Chen, Y.Z. (2000). *In homogeneous iterates of contraction mappings and nonlinear ergodtc theorem,* Nonlinear Anal., 39, pp. 1-10.

5. D. R. Smart (2003). Fixed Point Theorems, Cambridge Univ. Press.

6. Jha, K., Pant, R. P. and Singh, S. L. (2003). *Common fixed points for compatible mappings in metric space,* Radovi Mathematicki, 12, pp. 107-114.

7. Jungck G., Murthy, P. P., and Cho, Y. J. (1993). *Compatible mappings of type (A) and common fixed point theorem,* Math Japon. 38, pp. 381-390.

8. Kirk, William A., Brailey, S. (2001). *Handbook of Metric Fixed Point Theory. Springer-Verlag,* ISBN 0-7923-7073-2.

9. P. Kumlin (2003/2004). A note on Spectral Theory, Mathematics, Chalmers & GU

10. Pant, R. P. (1998). *Common fixed point theorem for contractive mappings,* J. Math. Ana. Appl., 226, pp. 257-258.

11. Pathak, H. K., Cho, Y. J., Kang, S. M., and Madharia, B. (1998). *Compatible mappings of type (C) and common fixed point theorems of Gergus type,* 30, pp. 499-518.

12. Sessa, S., and Fisher, B., (1989). *Common fixed point of weakly commenting mappings,* Bull. Acad. Polon. Sci. Ser. Sci. Math and Math. Sci., 12, pp. 147-156.

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