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A Comparative Review of Classical and Modified Periodogram used as the technique of estimating spectrum in Spectral Analysis

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Abstract – Periodogram is a non-parametric approach for estimating power spectrum in spectral analysis when sinusoids are fitted in discrete time series data. The performance of the Periodogram has always been measured based on spectral leakage, resolution, bias and variance. Thus from time to time, for getting better results from this technique, changes are made by various research workers and hence modified versions of Periodogram are introduced. In this paper reviews on original work of Schuster and Fourier periodograms are compared with the modified periodograms.

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INTRODUCTION

One of the simplest methods of determining the spectral content of a signal is through the use of discrete Fourier transformation (DFT) of its sampled sequence. The square of DFT points is an estimate of the power spectra density function (PSD). This is generally referred to as the Periodogram. Spectral analysis is the name given to methods of estimating the spectral density function, or spectrum, of a given time series. In the last century, research workers such as A. Schuster [9] were essentially concerned with looking for hidden periodicities in data, but spectral analysis as we know it today is mainly concern with the estimating the spectrum over the whole range of frequencies. The techniques are now widely used by many scientists, particularly in electrical engineering, physics, meteorology, marine science and in biomedical sciences etc.

In this paper we are explaining the original Periodogram which was firstly introduced by Schuster [9] and thus a classical Fourier Periodogram used by many researchers. Also we are explaining here, how its critique reduces the interest of various research fellows in its applications. To overcome its critic, how the various researchers modify this classical Fourier Periodogram into the various modified forms and these modified forms are known as modified periodograms. There are many modified Periodogram but in this paper we are explaining only Welch Periodogram [12] and Lomb-Scargle Periodogram [7]-[8] because of the reason that they are widely used in today's modern time. This paper has four sections 1-4, in first section Schuster Periodogram is described and in second section, Classical Fourier Periodogram is explained. In section third and fourth, Welch Periodogram and Lomb-Scargle Periodogram is explained with their advantages and disadvantages.

1. Schuster Periodogram:-

A. Schuster introduced the term 'Periodogram' in the discussion of Fourier analysis of empirical data. In some further papers (Schuster 1900,1906) he applied these ideas to the analysis of various set of data including sunspot series. According to him,

Let the time series numbers $t_1, t_2, t_3, \dots\dots$ are arranged in p intervals like as :-

$$t_1, t_2, \dots\dots\dots\dots\dots\dots\dots.., t_p$$

$$t_1', t_2', \dots\dots\dots\dots\dots\dots.. t_p'$$

$$\dots\dots\dots\dots\dots\dots$$

$$t_1^{s-1}, t_2^{s-1}, \dots\dots\dots\dots\dots\dots.. t_p^{s-1}$$

$$T_1 \quad T_2 \quad \dots\dots\dots\dots\dots\dots. \quad T_p$$

Where T_1, T_2 etc. are sums of vertical columns and T may be expressed by a periodic series of form:-

$$S = a_0 + a_1 cos\theta + a_2 cos2\theta + \cdots \ldots\ldots\ldots. + a_p cosp\theta + l$$
$$- 1)\theta$$

Where $S = T_1$ if we substitute $\theta = \frac{2\pi}{p}$ and generally S becomes T_q by the substitution of $\theta = \frac{2\pi q}{p}$. Now the coefficients are determined by a Fourier analysis

$$pa_0 = T_1 + T_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. + T_p$$

$$\frac{1}{2} pa_1 = T_1 cos\theta + T_2 cos2\theta + \cdots \ldots\ldots\ldots. + T_p cosp\theta$$

$$\frac{1}{2} pb_1 = T_1 sin\theta + T_2 sin2\theta + \cdots \ldots\ldots\ldots.. + T_p sinp\theta;\ \theta = \frac{2\pi}{p}$$

If there is a well marked periodicity corresponding to *p* intervals, let

$$r_1 = \sqrt{{a_1}^2 + {b_1}^2} \text{ and } \rho = \frac{r_1}{a_0}$$

as a measure of the amplitude of the periodicity corresponding to p intervals thus

$$\frac{\rho^2}{4} = \frac{{r^2}_1}{4{a^2}_0} = \frac{(\sum_i T_i \cos i\theta)^2}{\sum_i T_i} + \frac{(\sum_i T_i \sin i\theta)^2}{\sum_i T_i}$$

Schuster proposed the term Periodogram in the following way:-

$$\text{Let } \frac{1}{2} Ta = \int_{t_1}^{t_1+T} f(t) \cos kt \ dt, \qquad \frac{1}{2} Tb = \int_{t_1}^{t_1+T} f(t) \sin kt \ dt$$

Where T may for convenience be chosen to be equal to some integer multiple of $\frac{2\pi}{k}$, and plot a curve with $\frac{2\pi}{k}$ as abscissa and $r = \sqrt{a^2 + b^2}$ as ordinate, this curve represents the Periodogram of $f(t)$.

Beveridge (1921, 1922) gave an extensive Periodogram analysis of wheat price index at that time, he made a major computational venture. This method was used in second half of the nineteenth century to find periodic components of known periods in various types of time series data.

2. Classical Fourier Periodogram:-

A basic tool of spectral analysis is the discrete Fourier transform (DFT) which can be defined as an arbitrarily sampled data set $(X(t_j); j = 1,2, \ldots \ldots, N)$ as

$$FT_X(\omega) = \sum_{j=1}^{N} X(t_j)\, e^{-i\omega t_j} \text{ where } i = \sqrt{-1}$$

The Periodogram is then conventionally defined as

$$P_X(\omega) = \frac{1}{N} |FT_X(\omega)|^2$$
$$= \begin{cases} \frac{1}{N} \left|\sum_{j=1}^{N} X(t_j)\, e^{-i\omega t_j}\right|^2; \textit{for unevenspaced data} \\ \frac{1}{N} \left|\sum_{j=1}^{N} X(j)\, e^{-i\omega j}\right|^2; \textit{for evenlyspaced data} \end{cases}$$
$$= \begin{cases} \frac{1}{N} \left[\left(\sum_j X(t_j) \cos \omega t_j\right)^2 + \left(\sum_j X(t_j) \sin \omega t_j\right)^2\right]; \textit{for unevenspaced data} \\ \frac{1}{N} \left[\left(\sum_j X(j) \cos \omega j\right)^2 + \left(\sum_j X(j) \sin \omega j\right)^2\right]; \textit{for evenly spaced data} \end{cases}$$

The authors like Thompson (1971) [10] ; Deeming(1975) and many more called this function as classical Fourier Periodogram. This Periodogram works as if $X(t)$ contains a sinusoidal component of frequency ω_0 (say) then at and near to this value of frequency, the factors $X(t)$ and $\exp i\omega t$ would be in phase and will give a large contribution to the sums in above equation and at other values of ω the terms in the sum would be randomly positive and negative and result will be a small sum due to their cancellations. Hence, the presence of sinusoid is indicated by large value of $P_X(\omega)$. These equations can also be evaluated at any frequency but traditionally they are evaluated at a special set of $N_0 = \frac{N}{2}$ for evenly spaced frequencies.

This Periodogram has two serious problems, one is statistical difficulties and other is spectral leakage. These problems are explained as

1.1 $P_X(\omega)$ is very noisy even when the data is slightly noisy. Also noise is not reduced after increasing the sample size as increase in data size also increases the available frequencies in proportion.

1.2 Spectral leakage is simply that for a sinusoidal signal at a given frequency, the power in the Periodogram not only appears at ω but also leaks to other frequencies

Modified Periodograms:-

To overcome the above mentioned problems and also to reduce so many calculations in the work of Periodogram analysis, many research workers modify this classical Fourier Periodogram. Thus we get the modified periodograms. There are so many such modified versions of Periodogram but in this paper we are explaining only Welch Periodogram and Lomb-Scargle Periodogram due their widely applicability to the various engineering fields in modern times.

3. Welch Periodogram:-

The work of Cooley and Tukey (1965) [3] in calculating Fourier transforms of data as a technique of Fast Fourier transform (FFT) gave revolutionary changes in the spectral analysis techniques. As a application of FFT, Welch(1967) introduced a method to the estimation of power spectra which involves sectioning the data, taking modified Periodograms of these sections and then averaging these modified periodograms. This averaged Periodogram is known as Welch Periodogram.

Let $X(j), j = 0,1,2,3, \dots \dots . N-1$ be a sample from stationary sequence and let $X(j)$ has power spectral density $P(\omega), |\omega| \le \frac{1}{2}$. The segments, possibly overlapping, of length L with starting points of their segments D units apart. Let $X_1(j) = X(j), j = 0,1,2, \dots \dots, L-1$ be the first such segment then similarly $X_2(j) = X(j + D), j = 0,1,2, \dots \dots, L-1$ be the second one and finally $X_K(j) = X(j + (K-1)D), j = 0,1,2, \dots \dots, L - 1,$ thus these are K segments $\{X_1(j), X_2(j), \dots \dots \dots, X_K(j)\}$ which covers the entire record and $(K-1)D + L = N$

Welch suggested choosing either window from the following two types of windows:-

$$W_1(j) = 1 - \left[\frac{j - \frac{L-1}{2}}{\frac{L+1}{2}}\right]^2 \quad W_1(j) = 1 - \left|\frac{j - \frac{L-1}{2}}{\frac{L+1}{2}}\right| ; j = 0,1,2, \dots \dots \dots, L-1$$

And finite Fourier transforms of K segments are $FT_1(n), FT_2(n), \dots \dots \dots, FT_K(n)$ defined as:-

$FT_K(n) = \frac{1}{L}\sum_{j=0}^{L-1} X_K(j) W(j) e^{-2Kijn/L}$ and $i = \sqrt{-1}$; Finally K modified periodograms are defined as :-

$$I_p(\omega_n) = \frac{L}{U} \left|FT_p(n)\right|^2 ; p = 1,2, \dots \dots, K$$

Where $\omega_n = \frac{n}{L}; n = 0,1,2, \dots \dots, \frac{L}{2}$ and $U = \frac{1}{L}\sum_{j=0}^{L-1} W^2(j)$

The spectral estimate is the average of these periodograms i.e.

$$P(\omega) = \frac{1}{K}\sum_{p=1}^{K} I_p(\omega_n)$$

This function is known as Welch Periodogram.

Now, we are explaining how Welch showed that it overcomes the above mentioned problems of classical Fourier Periodogram:-

1.1 Welch proved that $Var(I_p(\omega_n)) = \frac{P^2(\omega_n)}{K}\left\{1 + 2\sum_{j=1}^{K-1}\frac{K-j}{K}\rho(j)\right\}$ where

$$\rho(j) = \left[\sum_{p=0}^{L-1} W(p)\, W(p + jD)\right]^2 \left[\sum_{p=0}^{L-1} W^2(p)\right]^2$$

Hence if there is averaging over K segments thus obtaining a reduction of the variance by a factor $\frac{1}{K}$.

1.2 Welch also proved that the time required to perform a finite Fourier transform on a sequence of length L is approximately $c'L \log_2 L$ where c' is a constant which depends upon the program and type of computer. Hence it requires the less computing time.

4. L-S Periodogram:-

Researchers (e.g. Barning1963,Wealau and Laung [11], Vanicek (1969) , Lomb(1975), Faulkner (1977)) used the least square fitting of sine waves of various periods to the data and thus provided the new approach of Periodogram.

Lomb(1975) introduced the technique of fitting sine waves by least square to unequally spaced astronomical data and thus produce a natural extension of Fourier methods to non-uniform data and also this method reduces to the Fourier power spectrum in the limit of equal spacing.

Scargle(1982) also emphasized on least square based Periodogram in the study of detection of periodic signal hidden in noise in astronomical data analysis.

Actually, Lomb and Scargle recognized independently problems associated with Periodogram analysis of unequally spaced data. Based on harmonic regression, they developed a least square power spectrum that fixes the invariance of time translation problem of the generalized power spectrum. Now, it is currently called as Lomb-Scargle Periodogram (LS Periodogram) which is defined as:-

$$P_X(\omega) = \frac{1}{2}\left\{\frac{\left[\sum_j X_j \cos\omega(t_j-\tau)\right]^2}{\sum_k \cos^2\omega(t_j-\tau)} + \frac{\left[\sum_j X_j \sin\omega(t_j-\tau)\right]^2}{\sum_j \sin^2\omega(t_j-\tau)}\right\}$$

where τ is defined by

$$\tan(2\omega\tau) = \frac{\left(\sum_j \sin 2\omega t_j\right)}{\left(\sum_j \cos 2\omega t_j\right)}$$

With this definition of Periodogram, Scargle showed the following results:-

1.1 Statistical behavior of the Periodogram for unevenly sampled data was identical to that for the case of even spacing.

1.2 The Periodogram analysis is exactly equivalent to least square fitting of sinusoids to the data.

1.3 Time translation invariance is retained in the Periodogram.

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