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Product of Topological Spaces

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Abstract* – *There are three important ways of creating new topological spaces from old ones. They are by forming "Subspaces", "Product Spaces" and quotient spaces".

Keyword* - *Arbitrary Collection, Cartesian Product, Project Mapping

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PRODUCT SPACES:-

1) Definition Cartesian Product of Two Sets:-

Let X_1, X_2 be any two non-empty sets, then the set $X_1 \times X_2$ defined by

$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$

is known as Cartesian product of X_1 and X_2.

2) Cartesian Product of n Sets:-

Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be n sets, then the set $X_1 \times X_2 \times \ldots\ldots\ldots \times X_n$ defined by

$X_1 \times X_2 \times \ldots\ldots\ldots \times X_n = \{(x_1, x_2, \ldots\ldots, x_n) : x_i \in X_i, i = 1, 2, \ldots\ldots, n\}$

is known as cartesian product of $X_1, X_2, \ldots\ldots, X_n$.

3) Cartesian product of arbitrary collection of sets:-

Let $\{X_\alpha : \alpha \in \Lambda\}$ be arbitrary collection of sets, then set $\{(\ldots\ldots, x_\alpha, \ldots\ldots) : x_\alpha \in X_\alpha, \forall \in \Lambda\}$ is the product of set of sets $\{X_\alpha : \alpha \in \Lambda\}$ and is written as $\prod_{\alpha\in\Delta} X_\alpha$ and x_α is known as the αth co-ordinate of $(\ldots\ldots, x_\alpha, \ldots\ldots)$ and is a member of X_α.

4) Projection mapping:-

Let $\{X_\alpha : \alpha \in \Lambda\}$ be arbitrary collection of sets. Let $X = \prod_{\alpha\in\Delta} X_\alpha$ be the product set of $\{X_\alpha : \alpha \in \Lambda\}$, then the mapping $P_\alpha : X \to X_\alpha$ defined by $P_\alpha(x) = x_\alpha$= αth co-ordinate of x, $\forall\ \alpha \in \Lambda$ are known as Projection mapping.

Note : (1) Projection mapping are onto mapping.

Proof. Let $X = \prod_{\alpha\in\Delta} X_\alpha$ and $P_\alpha : X \to X_\alpha$ be a projection mapping.

Let $x_\alpha \in X_\alpha$ be any point, then by definition of $X = \prod_{\alpha\in\Delta} X_\alpha$,

x_α is the α-th co-ordinate of atleast one member of X(say) $x \in X$.

$\therefore\ P_\alpha(x) = x_\alpha$

i.e. x is the pre-image of x_α under the mapping P_α

$\therefore$ P_α is onto mapping.

(2) Projection mapping are many-one in general.

Proof. Let $X = \prod_{\alpha\in\Delta} X_\alpha$, in general X_α contain more than one point $\forall\ \alpha \in \Lambda$.

Let $x_\alpha \neq y_\alpha$ be any two points of X_α, $\forall\ \alpha \in \Lambda$

$\therefore$ X contains more than one vector whose α-th co-ordinate is x_α.

$\therefore$ if we fix α-th co-ordinate = x_α and vary all other co-ordinate in X_β, $\alpha \neq \beta \in \Lambda$.

Then each vector in X will have α-th co-ordinate = x_α

$\therefore$ Each such vectors in X are mapped to same point x_α by the mapping

$P_\alpha : X \to X_\alpha$, $\forall\ \alpha \in \Lambda$

$\therefore$ Projection mapping are many-one in general.

5) Theorem.

Let $X = X_1 \times X_2$ and $x_1 \in X_1$, $x_2 \in X_2$ be fixed points.

Let $P_1 : X \to X_1$ and $P_2 : X \to X_2$ be projection mapping, then if

(i) $A_1 \subseteq X_1$, then $P_1^{-1}(A_1) = A_1 \times X_2$

(ii) $A_1 \subseteq X_2$, then $P_2^{-1}(A_2) = X_1 \times A_2$.

Proof. (i) Let $A_1 \subseteq X_1$ and let $y \in P_1^{-1}(A_1)$ be any point

=> $P_1(y) \in A_1$

=> First co-ordinate of $y \in A_1$

Let $y = (y_1, y_2)$, where $y_1 \in A_1$ and $y_2 \in X_2$

=> $y = (y_1, y_2) \in A_1 \times X_2$

=> $P_1^{-1}(A_1) \subseteq A_1 \times X_2$ ------------------- (1)

Again, let $z \in A_1 \times X_2$ be any point.

Let $z = (z_1, z_2)$, where $z_1 \in A_1$ and $z_2 \in X_2$

=> $P_1(z) = z_1 \in A_1$ => $P_1(z) \in A_1$ => $z \in P_1^{-1}(A_1)$

∴ $A_1 \times X_2 \subseteq P_1^{-1}(A_1)$ ------------------- (2)

From (1) and (2), we get,

$P_1^{-1}(A_1) = A_1 \times X_2$

(ii) Let $A_2 \subseteq X_2$, let $y \in P_2^{-1}(A_2)$ be any point

=> $P_2(y) \in A_2$

=> Second co-ordinate of $y \in A2$

Let $y = (y_1, y_2)$, where $y_1 \in X_1$ and $y_2 \in A_2$

=> $y = (y_1, y_2) \in X_1 \times A_2$

∴ $P_2^{-1}(A_2) \subseteq X_1 \times A_2$

Again, let $z \in X_1 \times A_2$ be any point.

Let $z = (z_1, z_2)$, where $z_1 \in X_1$, $z_2 \in A_2$

=> $P_2(z) = z_2 \in A_2$ => $P_2(z) \in A_2$ => $z \in P_2^{-1}(A_2)$

∴ $X_1 \times A_2 \subseteq P_2^{-1}(A_2)$

From (1) and (2), we get $P_2^{-1}(A_2) = X_1 \times A_2$

6) Theorem :

If $X = \prod_{\alpha\in\Lambda} X_\alpha$ and $A_\beta \subseteq X_\beta$, then

$P_\beta^{-1}(A_\beta) = \prod_{\alpha\in A} V_\alpha$, where $V_\alpha = \begin{cases} X_\alpha, & \forall\, \alpha \neq \beta \\ A_\beta, & \forall\, \alpha = \beta \end{cases}$

Proof. Let $x \in P_\beta^{-1}(A_\beta)$ be any point.

=> $P_\beta(x) \in A_\beta$

=> β-th co-ordinate of $x \in A_\beta$ i.e. $x_\beta \in A_\beta$

But $x_\alpha \in X_\alpha$, $\forall\, \alpha \in \Lambda$

∴ $x_\beta \in A_\beta$ and $x_\alpha \in X_\alpha$, $\forall\, \alpha \neq \beta$

∴ $x_\alpha \in V_\alpha$, $\forall\, \alpha \in \Lambda$, where $V_\alpha = \begin{cases} X_\alpha, & \text{if } \alpha \neq \beta \\ A_\beta, & \text{if } \alpha = \beta \end{cases}$

∴ $x_\alpha \in \prod_{\alpha\in\Lambda} V_\alpha$

and so $P_\beta^{-1}(A_\beta) \subseteq \prod_{\alpha\in\Lambda} V_\alpha$ ------------------------(1)

Again, let $y \in \prod_{\alpha\in\Lambda} V_\alpha$ be any point

=> $y_\alpha \in V_\alpha$, $\forall\, \alpha \in \Lambda$

In particular, $y_\beta \in V_\beta$

=> $y_\beta \in A_\beta$ => $P_\beta(y) \in A_\beta$ => $y \in P_\beta^{-1}(A_\beta)$

∴ $\prod_{\alpha\in\Lambda} V_\alpha \subseteq P_\beta^{-1}(A_\beta)$ ------------------------(2)

∴ From (1) and (2), we get

$P_\beta^{-1}(A_\beta) = \prod_{\alpha\in A} V_\alpha$.

7) Theorem.

Let $X = \prod_{\alpha\in A} X_\alpha$ and $A_{\alpha_i} \subseteq X_{\alpha_i}$, $\forall$ i = 1, 2, 3, ……, n,

Then $\cap_{i=1}^{n} P_{\alpha_i}^{-1}(A_{\alpha_i}) = \prod_{\alpha\in A} V_\alpha$, where

$V_\alpha = \begin{cases} X_\alpha : \forall\, \alpha \neq \alpha_i \text{ and } \forall\, i = 1,2,3,\ldots,n \\ A_\alpha : \forall\, \alpha = \alpha_1 \text{ or } \alpha_2 \text{ or} \ldots\ldots\ldots\, \alpha_n \end{cases}$

Proof. Let $\in \cap_{i=1}^{n} P_{\alpha_i}^{-1}(A_{\alpha_i})$, be any point

=> $x \in P_{\alpha_i}^{-1}(A_{\alpha_i})$, $\forall$ i = 1, 2, 3, ……, n

=> $P_{\alpha_i}(x) \in A_{\alpha_i}, \forall\, i = 1, 2, 3, \ldots\ldots, n$

=> $P_\alpha(x) \in A_\alpha, \forall\, \alpha = \alpha_1 \text{ or } \alpha_2 \text{ or} \ldots \ldots \text{or } \alpha_n$

=> $x_\alpha \in A_\alpha$, $\forall\, \alpha = \alpha_1$ or α_2 or………α_n

But $x_\alpha \in X_\alpha$, $\forall\, \alpha \in \Lambda$ always

and so $x_\alpha \in X$ for $\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_n$

$\therefore$ $x_\alpha \in V_\alpha$, where $V_\alpha = \begin{cases} X_\alpha : \forall\, \alpha \neq \alpha_I \text{ and } \forall\, i = 1,2,3,\ldots,n \\ A_\alpha : \forall\, \alpha = \alpha_1 \text{ or } \alpha_2 \text{ or} \ldots\ldots\ldots \alpha_n \end{cases}$

=> $x \in \prod_{\alpha\in\Lambda} V_\alpha$, where $V_\alpha = \begin{cases} X_\alpha : \forall\, \alpha \neq \alpha_I \text{ and } \forall\, i = 1,2,3,\ldots,n \\ A_\alpha : \forall\, \alpha = \alpha_1 \text{ or } \alpha_2 \text{ or} \ldots\ldots\ldots \alpha_n \end{cases}$

$\therefore$ $\cap_{i=1}^{n} P_{\alpha_i}^{-1}(A_{\alpha_i}) \subseteq \prod_{\alpha\in\Lambda} V_\alpha$ ---------------------------(1)

Again, $y = \prod_{\alpha\in\Lambda} V_\alpha$ let be any point

=> $y_\alpha \in V_\alpha$, $\forall\, \alpha \in \Lambda$

But $V_\alpha = A_\alpha$ for $\alpha = \alpha_1$ or α_2, or …….. or α_n

$\therefore$ $y_\alpha \in A_\alpha$ for $\alpha = \alpha_1$ or α_2 or………. or α_n

=> $y_{\alpha_i} \in A_{\alpha_i}$ for $\forall$ i =1, 2,…….., n

=> $P_{\alpha_i}(y) \in A_{\alpha_i}$ for $\forall$ i = 1, 2, …….., n

=> $y \in P_{\alpha_i}^{-1}(A_{\alpha_i})$ for $\forall$ i = 1, 2, ………, n

=> $y \in \cap_{i=1}^{n} P_{\alpha_i}^{-1}(A_{\alpha_i})$

$\therefore$ $\pi_{\alpha\in\Lambda} V_\alpha \subseteq \cap_{i=1}^{n} P_{\alpha_i}^{-1}(A_{\alpha_i})$ ----------------------(2)

From (1) and (2), we get

=> $\cap_{i=1}^{n} P_{\alpha_i}^{-1}(A_{\alpha_i}) = \prod_{\alpha\in\Lambda} V_\alpha$

Product Topological Spaces

Let $(X_\alpha, J_\alpha) : \alpha\in\Lambda$ be any topological spaces. Let $X = \prod_{\alpha\in\Lambda} X_\alpha$, be the cartesian product of $X_\alpha : \alpha \in \Lambda$ and $P_\alpha : X \to X_\alpha$ be the projection mapping.

Let Ş= $\{P_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathfrak{J}_\alpha, \forall \alpha \in \Lambda\}$

$= \{P_\alpha^{-1}(U_\alpha) : U_\alpha \text{ is open set in } X_\alpha, \forall \alpha \in \Lambda\}$

Clearly Ş ⊆ P(X), the power set of X

Also $X_\alpha \in \mathfrak{J}_\alpha\ \forall\, \alpha \in \Lambda$ => $P_\alpha^{-1}(X_\alpha) \in$ Ş

=> X ∈ Ş

$\therefore$ $\cup_{S\in Ş} S = X$

then, ∃ a **smallest topology** $\mathfrak{J}$ on X, which contain Ş and Ş is a sub-base for $\mathfrak{J}$ (see Th.) and this topology $\mathfrak{J}$ is called the product topology and (X, $\mathfrak{J}$) is called the product topology spaces of spaces $(X_\alpha, \mathfrak{J}_\alpha) : \alpha \in \Lambda$.

Remark. (i) Any base element for the product topology $\mathfrak{J}$ is equal to the intersection of finite number of members of Ş i.e. equal to

$$\bigcap_{\alpha\in F} P_\alpha^{-1}(U_\alpha) = \prod_{\alpha\in\Lambda} V_\alpha$$

where F is a finite subset of Λ and V_α is open set in X_α.

i.e. $V_\alpha = \begin{cases} U_\alpha & \text{if } \alpha \in F \text{ and } U_\alpha \text{ is open set in } X_\alpha \\ X_\alpha & \text{if } \alpha \notin F \end{cases}$

(ii) Projection mapping in product topology are continuous mapping

Since $\forall\, U_\alpha \in \mathfrak{J}_\alpha$, we have $P_\alpha^{-1}(U_\alpha) \in$ Ş

$\therefore$ $\forall\, U_\alpha \in \mathfrak{J}_\alpha$, we have $P_\alpha^{-1}(U_\alpha) \in \mathfrak{J}$

[∵ Ş ⊆ $\mathfrak{J}$]

$\therefore$ $P_\alpha : X \to X_\alpha$ is continuous mapping, $\forall\, \alpha \in \Lambda$.

1) Theorem.

Projection mapping are open mapping in product topological space.

Proof. Let $(X_\alpha, \mathfrak{J}_\alpha)$, $\alpha \in \Lambda$ be topological spaces and (X, $\mathfrak{J}$) be the product topological space.

Let $P_\alpha : X \to X_\alpha$ be the projection mapping, $\forall \alpha \in \Lambda$

Let G ∈ be any open set

$\therefore$ G = Union of some base elements for $\mathfrak{J}$

= $\cup_{i\in\Omega} B_i$, where B_i's are base element for $\mathfrak{J}$

Now $B_i = \prod_{\alpha\in\Lambda} V_\alpha$, where $V_\alpha = \begin{cases} U_\alpha & \text{if } \alpha \notin F_I \\ X_\alpha & \text{if } \alpha \notin F_i \end{cases}$

Where F_i is finite subset of Λ.

Now $P_\alpha(B_i) = V_\alpha$, $\forall\, \alpha \in \Lambda$

$\therefore$ $P_\alpha(B_i)$ is open set in X_α

=> $\cup_{i\in\Omega} P_\alpha(B_i)$ is open set in X_α

=> $P_\alpha(\cup_{i\in\Omega} B_i)$ is open set in X_α

=> $P_\alpha(G)$ is open set in X_α

∴ $\forall G \in \mathfrak{J}$ => $P_\alpha(G) \in \mathfrak{J}_\alpha$

and $P_\alpha : X \to X_\alpha$ so is open mapping.

Note : (1) Projection mapping in product topological space are continuous, open and onto mapping and not one-one in general. Therefore projection mapping are not homeomophism in general.

(2) Projection mapping are not necessarily closed mapping.

For example: Let (IR, U) be the usual topological space (IR^2, $\mathfrak{J}$) be the product space. Let $P : IR^2 \to IR$ be the projection mapping defined by $P(x, y) = x, \forall (x, y) \in IR^2$

Let F = {(x, y) : x, y ∈ IR s.t. xy = 1}

Then, it is easy to verify that F is a closed subset of IR^2.

But, P(F) = IR – {0}, which is not closed in (IR, U).

Hence, the projection mapping $P : IR^2 \to IR$ is not necessarily a closed mapping.

2) Theorem.

Let (X_α, $\mathfrak{J}_\alpha$), α ∈ Λ be topological spaces and (X, $\mathfrak{J}$) be the product topological space of (X_α, $\mathfrak{J}_\alpha$) ; α ∈ Λ. Let (Y, $\mathfrak{K}$) be any other topological space. Then the mapping *f* : Y → X is continuous iff P_α of is continuous, $\forall \alpha \in \Lambda$.

Proof. Firstly, let *f* : Y → X be continuous mapping. Also $P_\alpha : X \to X_\alpha$ be continuous mapping, $\forall \alpha \in \Lambda$.

Since composite of two continuous functions is continuous function. Therefore $P_\alpha \, o \, f : Y \to X_\alpha$ is also continuous function, $\forall \alpha \in \Lambda$.

Conversely let $P_\alpha \, o \, f : Y \to X_\alpha$ be continuous function, $\forall \alpha \in \Lambda$. To show that *f* : Y → X is also continuous.

Let G be any open set in X

∴ $G = \bigcup_{i \in \Omega} B_i$, where B_i's are base element for $\mathfrak{J}$

Now $B_i = \bigcap_{\alpha \in F_i} P_\alpha^{-1}(U_\alpha) \in \Lambda$, where F_i is a finite subset of Λ and U_α is open set in X_α.

Now U_α is open set in X_α, $\forall \alpha \in F_i$ and $P_\alpha \, o \, f : Y \to X_\alpha$ is continuous function.

∴ $(P_\alpha \, o \, f)^{-1}(U_\alpha)$ is open set in Y, $\forall \alpha \in F_i$

=> $f^{-1} o \, P_\alpha^{-1}(U_\alpha)$ is open set in Y, $\forall \alpha \in F_i$

=> $f^{-1}(P_\alpha^{-1}(U_\alpha))$ is open set in Y, $\forall \alpha \in F_i$

=> $\bigcap_{\alpha \in F_i} f^{-1}(P_\alpha^{-1}(U_\alpha))$ is open set in Y

=> $f^{-1}(\bigcup_{\alpha \in F_i} P_\alpha^{-1}(U_\alpha))$ is open set in Y

=> $f^{-1}(B_i)$ is open set in Y, $\forall i \in \Omega$

=> $\bigcup_{i \in \Omega} f^{-1}(B_i)$ is open set in Y

=> $f^{-1}(\bigcup_{i \in \Omega} B_i)$ is open set in Y

Thus, $\forall G \in \mathfrak{J}$ => $f^{-1}(G) \in \mathfrak{K}$

Hence, *f*: Y→X is also continuous mapping.

3) Theorem

Let (X, $\mathfrak{J}_1$) and (Y, $\mathfrak{J}_2$) be any two topological spaces and (X × Y, $\mathfrak{J}$) be their product space. Then for a fixed point x ∈ X, the subspace {x} × Y of (X × Y, $\mathfrak{J}$) is homeomorphic to (Y, $\mathfrak{J}_2$) and for a fixed point y ∈ Y, the subspace X × {y} is homeomorphic to (X, $\mathfrak{J}_1$).

Proof. We have that the projection mapping $P_y : X \times Y \to Y$ is continuous and open mapping.

Let $f_y = P_y \mid \{x\} \times Y$, the restriction of P_y to {x} × Y.

i.e. $f_y : \{x\} \times Y \to Y$ such that $f_y((x, y)) = y$, ∀ (x, y) ∈ {x} × Y

Then, *f* is one-one mapping, for

Let $f_y((x, y_1)) = f_y((x, y_2))$

=> $y_1 = y_2$

=> $(x, y_1) = (x, y_2)$

Also, for each y ∈ Y, ∃ an element (x, y) ∈ {x} × Y such that

$f_y((x, y)) = y$

∴ f_y is onto mapping.

Moreover f_y is a restriction of continuous mapping P_y

∴ f_y is also continuous.

Thus, in order to show that f_y is homeomophism, it remain to show that f_y is open mapping.

Let A be any subset of {x} × Y, open in {x} × Y. Then

A = ({x} × Y) ∩ B, for some $\mathfrak{I}$ – open subset B of X × Y.

Let B be a base for the product topology X × Y

∴ B can be expressed as the union of members of B

Let B = U{G × H : G ∈ $\mathfrak{I}_1$ and H ∈ $\mathfrak{I}_2$ for some G × H ∈ B}

∴ f_y(A) = f_y[({x} × Y) ∩ B]

= f_y [({x} × Y) ∩ (U {G × H : G ∈ $\mathfrak{I}_1$, H ∈ $\mathfrak{I}_1$

for some G × H ∈ B)]

= U { f_y [({x} ∩ G) × (Y ∩ H)] : G ∈ $\mathfrak{I}_1$, H ∈ $\mathfrak{I}_2$,

for some G × H ∈ B}

= U { f_y [({x} ∩ G) × H] : G ∈ $\mathfrak{I}_1$, H ∈ $\mathfrak{I}_2$

for some G × H ∈ B} ....(*)

[∵ H ⊆ Y ∴ H ∩ Y = H]

Now, x ∈ X be a fixed point of X and G ⊆ X

∴ if x ∉ G, then {x} ∩ G = ϕ, and so ({x} ∩ G) × H = ϕ

∴ f_y[({x} ∩ G × H] = $f_y(\phi)$ = ϕ

∴ from (*), we get,

f_y(A) = ϕ, which is $\mathfrak{I}_2$ – open set

and if , x ∈ G, then {x} ∩ G = {x}, and so

f_y(A) = U { f_y({x} × H) : H ∈ $\mathfrak{I}_2$}

= U {H:H ∈ $\mathfrak{I}_2$}

= arbitrary union of $\mathfrak{I}_2$ – open sets

= $\mathfrak{I}_2$ – open set

Thus, f_y is open mapping and so f_y is a homeomorphism

i.e. {x} × Y ≈ Y.

Similarly, for a fixed y ∈ Y, X × {y} is homomorphic of X

i.e. X × {y} ≈ X.

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