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# Solution of Some Diophantine Problems

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***Abstract – In this paper we shall study on the some famed history and integer solution of some Diophantine Equations in +ve integer solution and connection with Pythagorean triples and Pell's Equation. We have shown that how some solution can be found with the help of conversion into Diophantine equation.***

***Keywords: - Diophantine Equation, Pythagorean Triples, Pell's Equation, ErdöS- Strauss Conjuncture.***

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## INTRODUCTION

Equations with integer coefficients whose solutions are to be found in integers (or sometimes rational numbers) are called Diophantine equations in the honor of Diophantus of Alexandria (250 AD)[01]. The adjective 'Diophantine' pertains not so much to the nature of the equation as to the nature of the admissible solutions of the equation. Problems in Diophantine equations are easy to state but usually hard to solve. The difficulty arises due to the stringent restriction of admitting only integer solutions. Often it is difficult even to ascertain whether an integer solution exists or not. An extreme example is the famous Diophantine equation $x^n + y^n = z^n$, for arbitrary (n > 2), [02] The Erdos-Strauss conjuncture for every integer n ≥ 2, there exist a solution with x,y and z all positive integer, which is a special type of Diophantine equation[04]

### The Linear Diophantine Equation

A linear Diophantine equation is an equation of the first-degree whose solutions are restricted to integers. The prototypical linear Diophantine equation is:

$$ax + by = c \ldots\ldots\ldots\ldots\ldots (1)$$

Where a, b and c are integer coefficients and x and y are variables[05]. Typical linear Diophantine problems hence involve whole amounts, such as e.g.

*How many ways a man can buy apples and grapes with cost of Rs 5per kg apple and 25Rs. Per kg grapes so that total cost amount becomes equal to 200 rupees*.

The solution of this question can give the easily answer by this Diophantine equation

$$5x + 25y = 200. \ldots\ldots\ldots\ldots\ldots (2)$$

The greatest common divisor (G.C.D) of two or more integers, which are not all zero, is the largest positive integer that divides each of the integers. From above example, we can begin by factoring out the common divisor 5, obtaining: The greatest common divisor of a and b, i.e. 1 and 5, is 1. Any non-negative c is a multiple of 1. There are nine such multiples of 5 which are less than or equal to 40. They are 0, 5, 10, 15, 20, 25, 30, 35, and 40. Therefore, there are nine ways to buy apples and grapes in this ways. They are: (0, 8), (5, 7), (10, 6), (15, 5), (20, 4), (25, 3), (30, 2), (35, 1) or (40, 0).

### Pythagorean triples

The well-known Diophantine equation of all is a particular case of the equation from Fermat's Last Theorem, $x^n + y^n = z^n$, but for the n = 2and taking $x = a$, $y = b$, $z = c$ [06], therefore Equation becomes

$$a^2 + b^2 = c^2. \ldots\ldots\ldots\ldots\ldots (3)$$

This equation (3) gives the Pythagoras theorem of a right angle triangle of base, height and hypotenuse respectively a,b and c. This is the equation which helps to find the length of the sides of a right angled triangle. These three numbers (a,b,c) are called Pythagorean triples.

### Pell's Equation

John Pell (1611-1685) was an English mathematician who learned mathematics in Holland, at the universities of Amsterdam and Breda in 1640's. Pell's equation has a long interesting history from ancient India to Europe.

Pell's equation is an equation of the following form

$$x^2 - ny^2 = 1 \ldots\ldots\ldots \quad (4)$$

Where *n* is a given positive square-free integer & n > 1 so that *x* and y are integer solutions[03]. Obviously this is also a Diophantine nature equation.

To every Diophantine equation ax+by=c, there is another supplementary Diophantine equation

$$ax - by = {}^{1}/_{c}\,.$$

In equation (3) putting $x^2 = p$ and $y^2 = q$ and applying above statement, we get

$$p - nq = 1\,.\ldots\ldots\ldots \ (5)$$

And

$$p + nq = 1 \quad \ldots.\ldots\ldots \ (6)$$

Solving (5) and (6), we get

$$p = 1, \text{ from his} \quad x^2 = 1, \quad \text{therefore} \quad x = \pm 1 \ldots\ldots\ldots (7)$$

And

$$q = 0, \text{ From this} \quad y^2 = 0, \quad \text{therefore} \quad y = 0 \ \ldots\ldots\ldots (8)$$

Hence Pell's equation is nothing but a solution of Diophantine equation.

In Cartesian coordinates, the equation has the form of a hyperbola, as solutions to the equation occur wherever the curve passes through a point whose x and y coordinates are both integers, such as x = 1, y = 0 and x = -1, y = 0. Lagrange proved that as long as n is not a perfect square, Pell's equation has infinitely many distinct integer solutions.

**Solution of Diophantine equation**

We have the equation

$$4xyz = n(\,xy \,+\, xz \,+ yz\,) \ldots\ldots\ldots\ldots (9)$$

This particular equation solution comes up with arrangement into Diophantine formation, I would like to take some conclusions and remark here.

From equation (9)

$$4xyz = n(\,xy \,+ xz \,+\, \ yz\,)$$

Dividing both sides by $xyz$, we get

$$\Rightarrow \frac{4}{n} = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

$$\Rightarrow \left(\frac{1}{x} + \frac{1}{y}\right) = \left(\frac{4}{n} - \frac{1}{z}\right) \quad \ldots\ldots\ldots (10)$$

Let us take

$$p = \frac{1}{x}\,, \quad q = \frac{1}{y}$$

Therefore,

$$p + q = \left(\frac{4}{n} - \frac{1}{z}\right) = \frac{(4z - n\,)}{nz} \ldots\ldots\ldots.(11)$$

Now from Diophantine supplementary equation

$$p - q = \frac{nz}{(4z - n\,)} \qquad \ldots\ldots\ldots\ldots.(12)$$

Solving equation (11) and (12).

$$x = \frac{2nz(\,4z - n\,)}{(\,4z - n\,)^2 + \,n^2z^2} \qquad \ldots\ldots\ldots.(13)$$

and

$$y = \frac{2nz\,(4z - n\,)}{(4z - n\,)^2 - \,n^2z^2} \qquad \ldots\ldots\ldots.(14)$$

Equation (13) can be written as quadratics form in $z$, such that $f(z) = 0.$

$$x(4z \,-\, x^2)^2 + \,n^2z^2x = 2nz(4z - n) \qquad \ldots\ldots\ldots\ldots\ldots (15)$$

$$\text{Or} \quad z^2(16x + \,n^2x - 8x) \,+\, (2n^2 - 8nx) + \,n^2x = 0 \quad \ldots..(16)$$

Therefore from quadratic roots,

$$z = \frac{-(2n^2 - 8nx) \pm \sqrt{(2n^2 - 8nx)^2 - 4n^2x(16x + n^2x) - 8nx}}{2(16x + n^2x - 8n\,)} \ldots\ldots\ldots\ldots.(17)$$

$$z = \frac{-(8nx - 2n^2)\ \pm 2n^2\sqrt{1 - x^2}}{(32x + 2n^2x - 16n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Putting value

$$z = \frac{\left\{8n\left[\frac{2nz(4z-n)}{(4z-n)^2 + n^2z^2}\right] - 2n^2\right\} \pm 2n^2\sqrt{1 - \left[\frac{2nz(4z-n)}{(4z-n)^2+n^2z}\right]^2}}{\left\{32\left[\frac{2nz(4z-n)}{(4z-n)^2+n^2z^2}\right] + 2n^2\left[\frac{2nz(4z-n)}{(4z-n)^2+n^2z^2}\right] - 16n\,\right\}}$$

Or

$$z = \frac{\dfrac{\left\{16n^2z\,(4z-n) \pm 2n^2\sqrt{[(4z-n\,)^2 + n^2z^2]^2 - 4n^2z^2(4z-n)^2}\right\}}{(4z-n\,)^2 + n^2z^2}}{\dfrac{64nz(4z-n\,) + 2n^2(8nz^2 - 2n^2z\,) - 16n[(4z-n)^2 + n^2z^2]}{(4z-n\,)^2 + n^2z^2}}$$

$$z = \frac{16n^2z\,(4z-n) \pm 2n^2\sqrt{[(4z-n)^2 + n^2z^2]^2 - 4n^2z^2(4z-n)^2}}{64nz(4z-n) + 2n^2(8nz^2 - 2n^2z) - 16n[(4z-n)^2 + n^2z^2]} \quad \ldots\ldots\ldots\ldots(19)$$

When this is written in simplified form, we find

$$[32z^3 - 16n^2z^2 - 8nz + n^2]^2$$

$$= [(4z-n)^2 + n^2z^2]^2 - 2n^2z^2(4z-n)^2$$

This is now an equation of degree 6 of $f(z) = 0$, written as follow simple form,

$$1024z^6 - Xz^5 + Yz^4 + Zz^3 - Wz^2 + 32n^3z + 16z = 0 \ldots(20)$$

Here,

$$X = 1024 + 64n^2$$
$$Y = 32n^2 - 512n$$
$$Z = 256 + 256n - 56z^2 + 16n^3$$
$$W = 144n^3 + 32 - 2n^3$$

Since we have an idea about 6th degree equation has no any integral solution but has complex root solution.

For taking irrespective value of $n \geq 2$ we will get again an equation of degree 6. Hence there should be need of correction regarding value of $n$.

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