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A Research on the Fundamental Theorem of Calculus and Its Proof

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***Abstract** – The Fundamental Theorem of Calculus (FTC) and its proof provide an illuminating but also curious example. The propositional content of the statements, which are connected to this name, varies. Consequently, also the proofs differ. The formulations of different versions of "the" FTC cannot be understood in isolation from its historical context. In this paper we will formulate one of the most important results of calculus, the Fundamental Theorem. This result will link together the notions of an integral and a derivative. This paper contains a new elementary proof of the Fundamental Theorem of Calculus for the Lebesgue integral. The hardest part of our proof simply concerns the convergence in L1 of a certain sequence of step functions, and we prove it using only basic elements from Lebesgue integration theory.*

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INTRODUCTION

The fundamental theorem of calculus is a theorem that links the concept of differentiating a function with the concept of integrating a function. The first part of the theorem, sometimes called the first fundamental theorem of calculus, states that one of the antiderivatives (also called indefinite integral), say F, of some function f may be obtained as the integral off with a variable bound of integration. This implies the existence of antiderivatives for continuous functions. Conversely, the second part of the theorem, sometimes called the second fundamental theorem of calculus, states that the integral of a function f over some interval can be computed by using any one, say F, of its infinitely many antiderivatives. This part of the theorem has key practical applications, because explicitly finding the antiderivative of a function by symbolic integration avoids numerical integration to compute integrals. This provides generally a better numerical accuracy.

Although the Riemann sum is an important aspect of the integral concept, it is often tedious to evaluate an integral using the Riemann sum. For example, it requires a lot of effort to evaluate the integral of even a simple function, such as x^2 or $x^2 + x$, using the Riemann sum. The Fundamental Theorem of Calculus offers an elegant way of computing definite integrals without having to find limits of Riemann sums. The FTC acts as a bridge to connect the supposedly separate concepts of derivatives and definite integrals, concealing that differentiation and integration are, indeed, inverse processes.

The Fundamental Theorem of Calculus was developed independently by Sir Isaac Newton and Gottfried Leibniz. Newton pioneered his calculus between 1665 and 1667but did not publish it until 1687. On the other hand, Leibniz, who also discovered the same results in the mid-1670s, published his results before Newton, in 1684 and 1686. Leibniz treated the FTC mostly geometrically, whereas Newton viewed it dynamically, relating accumulation of a quantity and its rate of change. Although the core ideas of the FTC were first developed by Newton, most of the notation and terminology used in contemporary calculus textbooks and literature were those of Leibniz. The reason for this was that Newton's use of the terms fluents (variable quantities) and fluxions (the rate of change of such quanities), referring to integrals and derivatives, did not seem to appeal contemporary mathematicians. The modern packaged form of the FTC was due to du Bois-Reymond who put Cauchy's refined FTC in a combined form in 1876.

GEOMETRIC MEANING

For a continuous function $y = f(x)$ whose graph is plotted as a curve, each value of x has a corresponding area function $A(x)$, representing the area beneath the curve between 0 and x. The function $A(x)$ may not be known, but it is given that it represents the area under the curve.

The area under the curve between x and $x + h$ could be computed by finding the area between 0 and $x + h$, then subtracting the area between 0 and *x*. In other words, the area of this "strip" would be *A*(*x* + *h*) − *A*(*x*).

There is another way to *estimate* the area of this same strip. As shown in the accompanying figure, *h* is multiplied by *f*(*x*) to find the area of a rectangle that is approximately the same size as this strip. So:

$$A(x+h) - A(x) \approx f(x) \cdot h$$

In fact, this estimate becomes a perfect equality if we add the red portion of the "excess" area shown in the diagram. So: $A(x+h) - A(x) = f(x) \cdot h + (\text{Red Excess})$
Rearranging terms:

$$f(x) = \frac{A(x+h) - A(x)}{h} - \frac{\text{Red Excess}}{h}.$$

As *h* approaches 0 in the limit, the last fraction can be shown to go to zero. This is true because the area of the red portion of excess region is less than or equal to the area of the tiny black-bordered rectangle. More precisely,

$$\left|f(x) - \frac{A(x+h) - A(x)}{h}\right| = \frac{|\text{Red Excess}|}{h} \le \frac{h(f(x+h_1) - f(x+h_2))}{h} = f(x+h_1) - f(x+h_2),$$

where $x + h_1$ and $x + h_2$ are points where/reaches its maximum and its minimum, respectively, in the interval [x. x + *h]* By the continuity of*f* the latter expression tends to zero as *h* does. Therefore, the left-hand side tends to zero as *h* does, which implies

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}.$$

This implies f(x)=*A'(x).* That is. the derivative of the area function *A(x)* exists and is the original function *f(x):* so, the area function is simply an antiderivative of the original function. Computing the derivative of a function and "finding the area" under its curve are "opposite" operations. This is the crux of the Fundamental Theorem of Calculus.

FUNDAMENTAL THEOREM OF CALCULUS: PART I

Let *f(x)* be a bounded and continuous function on an interval [*a,* b].

Let $A(x) = \int_a^x f(t)\, dt.$

Then for $a < x < b,$

$$\frac{dA}{dx} = f(x).$$

In other words, this result says that $A(x)$ is an "antiderivative" of the original function, $f(x)^{13}$.

Example: an antiderivative -

Recall the connection between functions and their derivatives. Consider the following two functions:

$$g_1(x) = \frac{x^2}{2}, \quad g_2 = \frac{x^2}{2} + 1.$$

Clearly, both functions have the same derivative:

$$g_1'(x) = g_2'(x) = x.$$

We would say that $x^2/2$ is an "antiderivative" of *x* and that $(x^2/2) + 1$ is also an "an tiderivative" of a:. In fact, *any* function of the form

$$g(x) = \frac{x^2}{2} + C$$

where *C* is any constant

is also an "antiderivative" of *x.*

This example illustrates that adding a constant to a given function will not affect the value of its derivative, or, stated another way, antiderivatives of a given function are defined only up to some constant. We will use this fact shortly: if A(x) and *F(x)* are both antiderivatives of some function *f(x),* then *A(x)* = *F(x)* + *C.*

FUNDAMENTAL THEOREM OF CALCULUS: PART II

Let *f(x)* be a continuous function on [*a.b].* Suppose *F(x)* is *any* antiderivative of *f{x).* Then for $a \le x \le b,$

$$A(x) = \int_a^x f(t)\, dt = F(x) - F(a).$$

Proof-

From comments above, we know that a function *f(x)* could have many different antiderivatives that differ from one another by some additive constant. We are told that *F(x)* is an antiderivative of *f(x*). But from Part I of the Fundamental Theorem, we know that A(x) is also an antiderivative of *f{x).* It follows that

$$A(x) = \int_a^x f(t)\, dt = F(x) + C,$$

where *C* is some constant. (1)

However, by property-1 of definite integrals,

$$A(a) = \int_a^a f(t) = F(a) + C = 0.$$

Thus,

$$C = -F(a).$$

Replacing *C* by — *F(a)* in equation 1 leads to the desired result.

Thus $$A(x) = \int_a^x f(t)\,dt = F(x) - F(a).$$

THE DEFINITE INTEGRAL

We defined the definite integral, I, of a function $f(x) > 0$ on an interval [a, *b]* as the area under the graph of the function over the given interval We used the notation

$$I = \int_a^b f(x)dx \quad a \le x \le b.$$

to represent that quantity. We also set up a technique for computing areas: the procedure for calculating the value of *I* is to write down a sum of areas of rectangular strips and to compute a limit as the number of strips increases:

$$I = \int_a^b f(x)dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\Delta x, \quad (1)$$

where *N* is the number of strips used to approximate the region, *k* is an index associated with the *k'th* strip, and $\Delta x = x_{k+1} - x_k$ is the width of the rectangle. As the number of strips increases $(N \to \infty)$, and their width decreases $(\Delta x \to 0)$, the sum becomes a better and better approximation of the true area, and hence, of the definite integral, I.

We can generalize the definite integral to include functions that are not strictly positive, as shown in Figure 1. To do so, note what happens as we incorporate strips corresponding to regions of the graph below the *x* axis: These are associated with negative values of the function, so that the quantity $f(x_k)\Delta x$ in the above sum would be negative for each rectangle in the ' negative' portions of the function. This means that regions of the graph below the *x* axis will contribute negatively to the net value of *I.*

If we refer to A_1 as the area corresponding to regions of the graph of *f(x)* above the *x* axis, and A_2 as the total area of regions of the graph under the *x* axis, then we will find that the value of the definite integral I shown above will be

$$I = A_1 - A_2.$$

Thus the notion of "area under the graph of a function" must be interpreted a little carefully when the function dips below the axis.

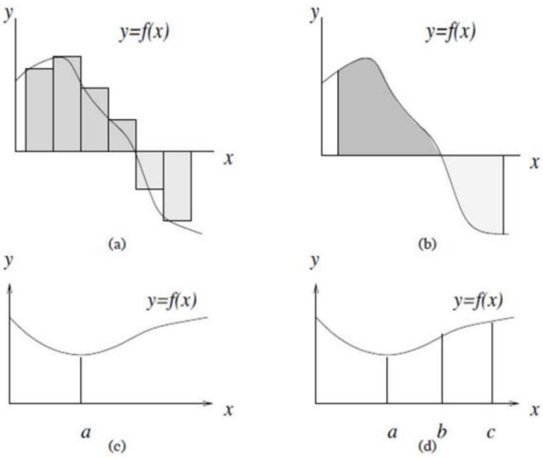

Figure 1. (a)If f(x) is negative in some regions, there are terms in the sum (1) that carry negative signs: this happens for all rectangles in parts of the graph that dip below the x axis. (b) This means that the definite integral $I = \int_a^b f(x)dx$ will correspond to the difference of two areas, A_1 — A_2 where A_1 is the total area (dark) of positive regions minus the total area (light) of negative portions of the graph. Properties of the definite integral: (c) illustrates Property 1.(d) illustrates Property 2.

PROOF OF THE FUNDAMENTAL THEOREM OF CALCULUS FOR THE LEBESGUE INTEGRAL

Let $f : [a, b] \longrightarrow \mathbb{R}$ be absolutely continuous on [a, b], i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that if is a family of pairwise disjoint subintervals of [a, *b]* satisfying $\{(a_j, b_j)\}_{j=1}^{n}$

$$\sum_{j=1}^{n} (b_j - a_j) < \delta$$

Then

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon.$$

Classical results ensure that f has a finite derivative almost everywhere in I = [a, b], and that $f' \in L^1(I)$. These results, which we shall use in this paper, are the first steps in the proof of the main connection between absolute continuity and Lebesgue integration: the Fundamental Theorem of Calculus for the Lebesgue integral.

Theorem 1 *If* $f : I = [a,b] \longrightarrow \mathbb{R}$ *is absolutely continuous on I then*

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$ *in Lebesgue's sense.*

In this note we present a new elementary proof to Theorem 1 which seems more natural and easy than the existing ones. Indeed, our proof can be sketched simply as follows:

1. We consider a well-known sequence of step functions $\{h_n\}_{n\in\mathbb{N}}$ which tends to f′ almost everywhere in I and, moreover, $\int_a^b h_n(x)\,dx = f(b) - f(a)$ for all $n \in \mathbb{N}.$

2. We prove, by means of elementary arguments, that $\lim_{n\to\infty} \int_a^b h_n(x)\,dx = \int_a^b f'(x)\,dx.$

In the sequel *m* stands for the Lebesgue measure in R.

Proof -

For each $n \in \mathbb{N}$ we consider the partition of the interval I = [a, b] which divides it into 2^n subintervals of length $(b-a)2^{-n},$ namely

$$x_{n,0} < x_{n,1} < x_{n,2} < \cdots < x_{n,2^n},$$

where

$x_{n,i} = a + i(b-a)2^{-n}$ for $i = 0, 1, 2, \ldots, 2^n.$

Now we construct a step function $h_n : [a,b) \longrightarrow \mathbb{R}$ as follows: for each $x \in [a,b)$ there is a unique $i \in \{0, 1, 2, \ldots, 2^n - 1\}$ such that $x \in [x_{n,i}, x_{n,i+1}),$

and we define

$$h_n(x) = \frac{f(x_{n,i+1}) - f(x_{n,i})}{x_{n,i+1} - x_{n,i}} = \frac{2^n}{b-a}[f(x_{n,i+1}) - f(x_{n,i})].$$

On the one hand, the construction of $\{h_n\}_{n\in\mathbb{N}}$ implies that $\lim_{n\to\infty} h_n(x) = f'(x)$ for all $x \in [a,b) \setminus N,$ (1)

Where $N \subset I$ is a null-measure set such that $f'(x)$ exists for all $x \in I \setminus N.$

On the other hand, for each $n \in \mathbb{N}$ we compute

$$\int_a^b h_n(x)dx = \sum_{i=0}^{2^n-1} \int_{x_{n,i}}^{x_{n,i+1}} h_n(x)dx = \sum_{i=0}^{2^n-1} [f(x_{n,i+1}) - f(x_{n,i})] = f(b) - f(a),$$

and therefore it only remains to prove that

$$\lim_{n\to\infty} \int_a^b h_n(x)\,dx = \int_a^b f'(x)\,dx.$$

Let us prove that, in fact, we have convergence in $L^1(I),$ i.e.,

$$\lim_{n\to\infty} \int_a^b |h_n(x) - f'(x)|\,dx = 0. \quad (2)$$

Let $\varepsilon > 0$ be fixed and let $\delta > 0$ be one of the values corresponding to $\varepsilon/4$ in the definition of absolute continuity of f.

Since $f' \in L^1(I)$ we can find $\rho > 0$ such that for any measurable set $E \subset I$ we have $\int_E |f'(x)|\,dx < \frac{\varepsilon}{4}$ whenever $m(E) < \rho.$ (3)

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