

Fundamental Groups of Homology Theory

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Abstract – Homology is a central concept for Developmental Evolution. Here I argue that homology have to be defined within the reference processes of improvement and evolution; development due to the fact it's miles the proximate reason of morphological characters and evolution because it deals with natural alterations and balance. This changed into already identified via Hans Spemann in 1915. In a seminal essay "A history and critique of the homology idea" Spemann analyzed the history and gift troubles of the homology concept. Here I will preserve Spemann's venture and analyze a number of the 20th century contributions to homology. I will stop with a few reflections approximately the connections among developmental methods and homology and conclude that developmental strategies are inherent in (i) the evaluation of homology, (ii) the explanation of homology, (iii) the beginning of evolutionary improvements (incipient homologues), and (iv) may be considered homologous themselves. iN

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INTRODUCTION

Homology, in mathematics, a fundamental belief of algebraic topology. Intuitively, two curves in an aircraft or other n -dimensional surface are homologous if together they certain a location—thereby distinguishing between an internal and an outdoor. Similarly, two surfaces inside a three-dimensional area are homologous if together they sure a three-dimensional place lying in the ambient area.

There are many ways of creating this intuitive notion specific. The first mathematical steps had been taken within the 19th century by way of the German Bernhard Riemann and the Italian Enrico Betti, with the introduction of "Betti numbers" in each dimension, relating to the number of unbiased (certainly described) gadgets in that size that are not boundaries. Informally, Betti numbers talk to the quantity of instances that an item can be "reduce" earlier than splitting into separate pieces; as an instance, a sphere has Betti quantity 0 on the grounds that any reduce will split it in, even as a cylinder has Betti no 1 considering that a reduce along its longitudinal axis will merely result in a rectangle. A extra huge remedy of homology turned into carried out in n dimensions at the start of the twentieth century by means of the French mathematician Henri Poincaré, main to the perception of a homology group in each measurement, seemingly first formulated approximately 1925 by way of the German mathematician Emmy No ether. The two

fundamental information approximately homology organizations for a surface or a better-dimensional topological manifold are: (1) if the businesses are described with the aid of a triangulation, a mobile subdivision, or different artifact, the ensuing businesses do now not depend on the particular alternatives made alongside the way; and (2) the homology companies are a topological invariant, in order that if two surfaces or better-dimensional spaces are homeomorphic, then their homology organizations in every measurement are isomorphic (see foundations of mathematics: Isomorphic structures and arithmetic: Algebraic topology).

Homology performs a essential position in analysis; certainly, Riemann become brought about it via questions related to integration on surfaces. The simple cause is because of Green's theorem (see George Green) and its generalizations, which express positive integrals over a website in terms of integrals over the boundary. As a consequence, certain vital integrals over curves may have the identical value for any curves which might be homologous. This is in flip pondered in physics in the examine of conservative vector spaces and the existence of potentials.

REVIEW OF LITERATURE

Stillwell 1993, In mathematics, homology is a popular way of associating a series of algebraic items such as abelian organizations or modules to other mathematical items which include topological

spaces. Homology agencies had been at the start described in algebraic topology. Similar structures are available in a wide sort of other contexts, such as summary algebra, businesses, Lie algebras, Galois Theory, and algebraic geometry.

The original motivation for outlining homology agencies become the observation that two shapes may be outstanding by examining their holes. For instance, a circle is not a disk due to the fact the circle has a hollow via it even as the disk is stable, and the ordinary sphere is not a circle due to the fact the sphere encloses a -dimensional hole while the circle encloses a one-dimensional hollow. However, because a hollow is "not there", it is not immediately obvious how to outline a hollow or how to differentiate distinctive forms of holes. Homology became at the beginning a rigorous mathematical method for outlining and categorizing holes in a manifold. Loosely talking, a cycle is a closed sub manifold, a boundary is a cycle which is also the boundary of a sub manifold, and a homology elegance (which represents a hole) is an equivalence elegance of cycles modulo boundaries. A homology class is thus represented by a cycle which isn't the boundary of any sub manifold: the cycle represents a hollow, particularly a hypothetical manifold whose boundary could be that cycle, but which is "not there".

There are many exclusive homology theories. A particular kind of mathematical object, consisting of a topological area or a group, can also have one or more associated homology theories. When the underlying item has a geometrical interpretation as topological spaces do, the n th homology group represents behavior in dimension n . Most homology agencies or modules may be formulated as derived elements on suitable abelian classes, measuring the failure of a component to be genuine. From this abstract attitude, homology agencies are decided by way of items of a derived category.

Weibel (1999) Homology idea can be stated first of all the Euler polyhedron formula, or Euler function. This was followed by Riemann's definition of genus and n -fold connectedness numerical invariants in 1857 and Betti's proof in 1871 of the independence of "homology numbers" from the choice of basis.

Richeson (2008) Homology itself became developed as a way to examine and classify manifolds consistent with their cycles – closed loops (or greater usually sub manifolds) that may be drawn on a given n dimensional manifold but now not continuously deformed into every different. These cycles are also every so often idea of as cuts which may be glued again together, or as zippers which may be mounted and loose. Cycles are classified through measurement. For instance, a line drawn on a surface represents a 1-cycle, a closed loop or (1-manifold), whilst a surface reduces thru a three-dimensional manifold is a 2-cycle.

RESEARCH METHODOLOGY

Covering Spaces of a Graph

We define here the notion of linear graph and prove the basic theorem that any covering space of a linear graph is itself a linear graph.

Recall that an arc A is a space homeomorphic to the unit interval $[0, 1]$. The end points of A are the points p, q corresponding to 0 and 1 under the homeomorphism; they are the unique points of A such that $A - p$ and $A - q$ are connected. The interior of an area A consists of A with its end points deleted..

Definition 1:- A linear graph is a space X that is written as the union of a collection of subspaces A_α , each of which is an arc, such that:

- (1) The intersection $A_\alpha \cap A_\beta$ of two arcs is either empty or consists of a single point that is an end point of each.
- (2) The topology of X is coherent with the subspaces A_α .

The arcs A_α are called the edges of X , and their interiors are called the open edges of X . Their end points are called the vertices of X ; we denote the set of vertices of X by X^0 .

If X is a linear graph, and if C is a subset of X that equals a union of edges and vertices of X , then C is closed in X . For the intersection of C with A_α is closed in A_α , since it is either empty, or it equals A_α , or it equals one or both vertices of A_α . It follows that each edge of X is a closed subset of X . It also follows that X^0 is a closed discrete subspace of X , since any subset of X^0 is closed in X .

In the case of finite graph, considered earlier, we used the Hausdorff condition in our definition in place of condition (2); it followed, in that case, that the topology of X was coherent with the subspaces A_α . In the case of an infinite graph, this would no longer be true, so we must assume the coherence condition as part of the definition. We would assume the Hausdorff condition as well, but it is no longer necessary, for it follows from the coherence condition:

Lemma 1:- Every linear graph X is Hausdorff; in fact, it is normal.

Proof. Let B and C be disjoint closed subsets of X . Assume, without loss of generality, that every vertex of X belongs either to B or to C . For each α , choose disjoint subsets U_α and V_α of A_α that are open in A_α , containing $B \cap A_\alpha$ and $C \cap A_\alpha$,

respectively. Let $U = \bigcup U_\alpha$ and $V = \bigcup V_\alpha$. Then U and V contain B and C , respectively.

We show the sets U and V are disjoint. If $x \in U \cap V$, then $x \in U_\alpha \cap V_\beta$ for some $\alpha \neq \beta$. This fact implies that A_α and A_β contain the point x , which means that x is a vertex of X . This is impossible, for if $x \in B$, the x lies in no set V_β , and if $x \in C$, then x lies in no set U_α .

Now we show U and V are open in X . To show U is open, we show that $U \cap A_\alpha = U_\alpha$ for each α . By definition, $U \cap A_\alpha$ contains U_α . If x is a point of $U \cap A_\alpha$ not in U_α , then x belongs to U_β for some $\beta \neq \alpha$. Then both A_β and A_α contain x , so that x must be a vertex of X . This is impossible, for if $x \in B$, then $x \in U_\alpha$ definition of U_α , and if $x \in C$, then x cannot belong to U .

Definition 2 :- Let X be a linear graph. Let Y be a subspace of X that is a union of edges of X . Then Y is closed in X and is itself a linear graph; we call it subgroup of X .

To show that Y is a linear graph, we need to show that the subspace topology on Y is coherent with the set of edges of Y . If the subset D of Y is closed in the subspace topology, then D is closed in X , so that $D \cap A_\alpha$ is closed in A_α for each edge of X , and in particular for each edge of Y . Conversely, suppose $D \cap A_\beta$ is closed in A_β for each edge A_β of Y . We must show that $D \cap A_\alpha$ is closed in A_α for each edge A_α of X that is not contained in Y . But in this case, $D \cap A_\alpha$ is either empty or a one-point set. We conclude that Y has the topology coherent with its set of edges.

Lemma 2:- Let X is a linear graph. If C is a compact subspace of X , there exists a finite sub graph Y of X that contains C . If C is connected, Y can be chosen to be connected.

Proof. First, note that C contains only finitely many vertices of X . For $C \cap X^0$ is a closed discrete subspace of the compact space C ; since it has no limit point, it must be finite. Similarly, there are only finitely many values of α for which C contains an interior point of the edge A_α . For if we choose a point x_α of C interior to A_α for each index α for which it is possible to do so, we obtain a collection $B = \{x_\alpha\}$ whose intersection with each edge A_β is a one-point set or empty. It follows that every subset of B is closed in X , so that B is a closed discrete subspace of C and hence finite.

Form Y by choosing, for each vertex x of X belonging to C , an edge of X having x as a vertex, and adjoining to these edges all edges A_α whose interiors contain points of C . They Y is a finite sub graph containing C . Note that if C is connected, then Y is the union of a collection of arcs each of which intersects C , so that Y is connected.

Lemma 3:- If X is a linear graph, then X is locally path connected and semi locally simply connected.

Proof. Step 1. We show X is locally path connected. If $x \in X$ and x lies interior to some edge of X , then within every neighborhood of x is a neighborhood of x homeomorphic to an open interval of \mathbb{R} , which is path connected. On the other hand, if x is a vertex of X and U is a neighborhood of x , then we can choose, for each edge A_α having x as an end point, a neighborhood V_α of x in A_α lying in U that is homeomorphic to the half-open interval $[0, 1)$. Then $\bigcup V_\alpha$ is a neighborhood of x in X lying in U , and it is a union of path-connected spaces having the point x is common.

Step 2. We show X is semi locally simply connected. Indeed, we show that if $x \in X$, the x has a neighborhood U such that $\pi_1(U, x)$ is trivial.

If x lies interior to some edge of X , then the interior of this edge is such a neighborhood. So suppose x

is a vertex of X . Let $\overline{St} x$ denote the union of those edges of X that have x as an end point, and let $St x$

denote the subspace of $\overline{St} x$ obtained by deleting all vertices other than x . ($St x$ is called the star of x in X). The set $St x$ is open in X , since its complement is a union of arcs and vertices. We show that $\pi_1(St x, x)$ is trivial.

Let f be a loop in $St x$ based at x . Then the image set $f(1)$ is compact, so it lies in some finite union of

arcs of $\overline{St} x$. Any such union is homeomorphic to the union of a finite set of line segments in the plane having an end point in common. And for any loop in such a space, the straight-line homotopy will shrink it to the constant loop at x .

Theorem 1 :- Let $p : E \rightarrow X$ be a covering map, where X is a linear graph. If A_α is an edge of X and B is a path component of $p^{-1}(A_\alpha)$, then p maps B homeomorphically onto A_α . Furthermore, the space E is a linear graph, with the path components of the spaces $p^{-1}(A_\alpha)$ as its edges.

Proof. Step 1. We show that p maps B homeomorphically onto A_α . Because the arc A_α is path connected and locally path connected. Since the map $p_0 : B \rightarrow A_\alpha$ obtained by restricting p is a covering map. Because B is path connected, the lifting correspondence $\phi : \pi_1(A_\alpha, a) \rightarrow P_0^{-1}(a)$ is surjective; because A_α is simply connected, $P_0^{-1}(a)$ consists of a single point. Hence p_0 is a homeomorphism.

Step 2. Since X is the union of the arcs A_α , the space E is the union of the arcs B that are the path

components of the space $p^{-1}(A_\alpha)$. Let B and B' be path components of $p^{-1}(A_\alpha)$ and $p^{-1}(A_\beta)$, respectively, with $B \neq B'$. We show B and B' intersect in at most a common end point. If A_α and A_β are equal, then B and B' are disjoint, and if A_α and A_β are disjoint, so are B and B' . Therefore, if B and B' intersect, A_α and A_β must intersect in an end point x of each; then $B \cap B'$ consists of a single point, which must be an end point of each.

Step 3. We show that E has the topology coherent with the arcs B . This is the hardest part of the proof. Let W be a subset of E such that $W \cap B$ is open in B , for each arc B of E . We show that W is open in E .

First, we show that $p(W)$ is open in X . If A_α is an edge of X , then $p(W) \cap A_\alpha$ is the union of the sets $p(W \cap B)$, as B ranges over all path components of $p^{-1}(A_\alpha)$. Each of these sets $p(W \cap B)$ is open in A_α , because p maps B homeomorphically onto A_α ; hence their union $p(W) \cap A_\alpha$ is open in A_α . Because X has the topology coherent with the subspaces A_α , the set $p(W)$ is open in X .

Second, we prove our result in the special case where the set W is contained in one of the slices V of $p^{-1}(U)$, where U is an open set of X that is evenly covered by p . By the result just proved, we know that the set $p(W)$ is open in X . It follows that $p(W)$ is open in U . Because the map of V onto U obtained by restricting p is a homeomorphism, W must be open in V , and hence open in E .

Finally, we prove our result in general. Choose a covering A of X by open sets U that are evenly covered by p . Then the slices V of the sets $p^{-1}(U)$, for $U \in A$, cover E . For each such slice V , let $W_V = W \cap V$. The set W_V has the property that for each arc B of E , the set $W_V \cap B$ is open in B , for $W_V \cap B = (W \cap B) \cap (V \cap B)$ and both $W \cap B$ and $V \cap B$ are open in B . The result of the preceding paragraph implies that W_V is open in E . Since W is the union of the sets W_V , it also is open in E .

DATA ANALYSIS

The Fundamental Group of a Graph

Now we prove the basic theorem that the fundamental group of any linear graph is a free group. Henceforth we shall refer to a linear graph simply as a graph.

Definition 3 An oriented edge e of a graph X is an edge of X together with an ordering of its vertices; the first is called the initial vertex, and the second, the final vertex, of e . An edge path in X is a sequence e_1, \dots, e_n of oriented edges of X such that the final vertex of e_i equals the initial vertex of e_{i+1} , for $i = 1, \dots, n - 1$. Such an edged path is entirely specified by the sequence of vertices x_0, \dots, x_n ,

where x_0 is the initial vertex of e_1 and x_i is the final vertex of e_i for $i = 1, \dots, n$. It is said to be an edge path from x_0 to x_n . It is called a closed edge path if $x_0 = x_n$.

Given an oriented edge e of X , let f_e be the positive linear map of $[0, 1]$ onto e ; it is a path from the initial point of e to the final point of e . Then, corresponding to the edge path e_1, \dots, e_n from x_0 to x_n , one has the actual path

$$f = f_1 * (f_2 * (\dots * f_n))$$

from x_0 to x_n , where $f_i = f_{e_i}$; it is uniquely determined by the edge path e_1, \dots, e_n . We call it the path corresponding to the edge path e_1, \dots, e_n . If the edge path is closed, then the corresponding path f is a loop.

A graph X is connected if and only if every pair of vertices of X can be joined by an edge path in X .

Proof. Suppose X is connected. Define $x \sim y$ if there is an edge path in X from x to y . For any edge of X , its end points belong to the same equivalence class; let Y_x denote the union of all edges whose end points are equivalent to x . Then Y_x is a subgraph of X and hence is closed in X . The subgraphs Y_x form a partition of X into disjoint closed subspaces; since X is connected, there must be only one such.

Conversely, suppose every pair of vertices of X can be joined by an edge path. Then they can be joined by an actual path in X . Hence all the vertices of X belong to the same component of X . Since each edge is connected, it also belongs to this component. Thus X is connected.

Definition 4:- Let e_1, \dots, e_n be an edge path in the linear graph X . It can happen that for some i , the oriented edges e_i and e_{i+1} consist of the same edge of X , but with opposite orientations. If this situation does not occur, then the edge path is said to be a reduced edge path.

Note that if this situation does occur, then one can delete e_i and e_{i+1} from the sequence of oriented edges and still have an edge path remaining (provided the original sequence consists of at least three edges). This deletion process is called reducing the edge path. It enables one to show that in any connected graph, every pair of distinct vertices can be joined by a reduced edge path.

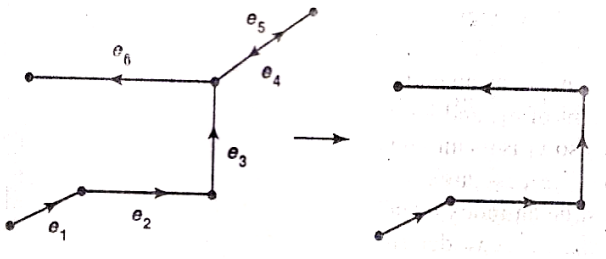


Figure 1

Definition 5: A sub graph T of X is said to be a tree in X if T is connected and T contains no closed reduced edge paths.

A linear graph consisting of a single edge is a tree. The graph in Figure.2 is not a tree, but deletion of the edge e would make it a tree. The graph in Figure 3 is a tree; deletion of the edge A would leave a tree remaining.

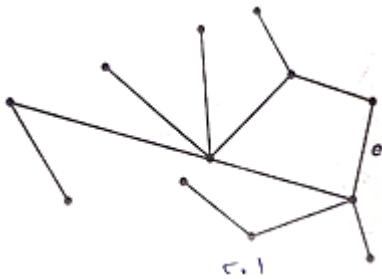


Figure.2

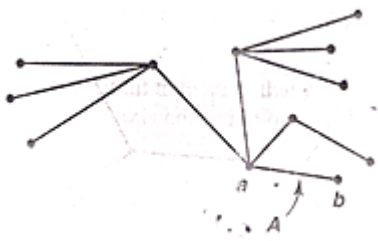


Figure.3

Theorem 2 :- Any tree T is simply connected.

Proof. We first consider the case where T is a finite tree. If T consists of a single edge, then T is simply connected. If T has n edges with $n > 1$, there is an edge A of T such that $T = T_0 \cup A$, where T_0 is a tree with $n - 1$ edges and $T_0 \cap A$ is a single vertex. Then T_0 is a deformation retracts of T . Since T_0 is simply connected by the induction hypothesis, so is T .

To prove the general case, let f be a loop in T . The image set of f is compact and connected, so it is contained in a finite connected sub graph Y of T . Now Y contains no closed reduced edge paths, because T contains none. Thus Y is a tree. Since Y is finite, it is simply connected. Hence f is path homotopic to a constant in Y .

Definition 6:- A tree T in X is maximal if there is no tree in X that properly contains T .

Theorem 3 :- Let X be a connected graph. A tree T in X is maximal if and only if it contains all the vertices of X .

Proof. Suppose T is a tree in X that contains all the vertices of X . If Y is a subgraph of X that properly contains T , we show that Y contains a closed reduced edge path; it follows that T is maximal. Let A be an edge of Y that is not in T ; by hypothesis, the end points a and b of A belong to T . Since T is connected, we can choose a reduced edge path e_1, \dots, e_n in T from a to b . If we follow this sequence by the edge A , oriented from b to a , we obtain closed reduced edge path in Y .

Now let T be a free in X that does not contain all the vertices of X . We show T is not maximal. Let x_0 be a vertex of X not in T . Since X is connected, we may choose an edge path in X from x_0 to a vertex of T , specified by the sequence of vertices x_0, \dots, x_n . Let i be the smallest index such that $x_i \in T$. Let A be the edge of X with vertices x_{i-1} and x_i . Then $T \cup A$ is a tree in X , by the preceding lemma, and $T \cup A$ properly contains T .

Theorem 4 :- If X is a linear graph, every tree T_0 in X is contained in a maximal tree in X .

Proof. We apply Zorn's lemma to the collection \mathcal{T} of all trees in X that contains T_0 , strictly partially ordered by proper inclusion. To show this collection has a maximal element, we need only prove the following:

If \mathcal{T}' is a sub collection of \mathcal{T} that is simply ordered by proper inclusion, then the union Y of the elements of \mathcal{T}' is a tree in X .

To begin, we note that since Y is a union of sub graphs of X , it is a sub graph of X . Second, since Y is a union of connected spaces that contain the connected space T_0 , the space Y is connected.

Finally, we suppose that e_1, \dots, e_n is a closed reduced edge path in Y and derive a contradiction. For each i , choose an element T_i of \mathcal{T}' that contains e_i . Because \mathcal{T}' is simply ordered by proper inclusion, one of the trees T_1, \dots, T_n , say T_j , contains all the others. But then e_1, \dots, e_n is a closed reduced edge path in T_j , contrary to hypothesis.

CONCLUSION

Subgroups of Free Groups

We now prove our main theorem, to the effect that a subgroup H of a free group F is free. The method of proof, remarkably enough, will give us some information about the cardinality of a system of free generator for H , when the cardinality of a system of free generators for F is known.

Theorem 5:- If H is a subgroup of a free group F , then H is free.

Proof. Let $\{\alpha \mid \alpha \in J\}$ be a system of free generators for F . Let X be a wedge of circles S_α , one for each $\alpha \in J$; let x_0 be their common point. We can give X the structure of a linear graph by breaking each circle S_α into three arcs, two of which have x_0 as an end point. The function that assigns to each α , a loop generating $\pi_1(S_\alpha, x_0)$, induces an isomorphism of F with $\pi_1(X, x_0)$. Therefore we may as well assume that F equals the group $\pi_1(X, x_0)$.

The space X is path connected, locally path connected, and semi locally simply connected. Therefore applies to show that there exists a path-connected covering space $p : E \rightarrow X$ of X such that, for some point $e_0 \in p^{-1}(x_0)$.

$$p_*(\pi_1(E, e_0)) = H.$$

Since p_* is a monomorphism, $\pi_1(E, e_0)$ is isomorphic to H .

The space E is a linear graph by. Then it implies that its fundamental group is a free group.

Definition 7 :- If X is a finite linear graph, we define the Euler number of X to be the number of vertices of X minus the number of edges. It is commonly denoted by the Greek letter chi, as $\chi(X)$.

Lemma 4 :- If X is a finite, connected linear graph, then the cardinality of a system of free generators for the fundamental group of X is $1 - \chi(X)$.

Proof.

Step 1. We first show that for any finite tree T , we have $\chi(T) = 1$. We proceed by induction on the number n of edges in T . If $n = 1$, then T has one edge and two vertices, so $\chi(T) = 1$. If $n > 1$, we can write $T = T_0 \cup A$, where T_0 is a tree having $n - 1$ edges, and A is an edge that intersects T_0 in a single vertex. We have $\chi(T_0) = 1$ by the induction hypothesis. The graph T has one more edge and one more vertex than T_0 ; hence $\chi(T) = \chi(T_0)$.

Step 2. We prove the theorem. Given X , let T be an maximal tree in X . If $X = T$, we are finished.

Otherwise, let A_1, \dots, A_n be the edges of X that are not in T . Then the fundamental group of X has a system of n free generators. On the other hand, X and T have exactly the same vertex set, and X has n more edges than T . Hence,

$$\chi(X) = \chi(T) - n = 1 - n.$$

So that $n = 1 - \chi(X)$

Definition 8 :- Let H be a subgroup of the group G . If the collection G/H of right cosets of H in G is finite, its cardinality is called the index of H in G . (The collection of left cosets of H in G has the same cardinality, of course).

Theorem 6 :- Let F be a free group with $n + 1$ free generators; let H be a subgroup of F . If H has index k in F , then H has $kn + 1$ free generators.

Proof. We apply the construction given in the proof of previous theorem. We can assume that $F = \pi_1(X, x_0)$, where X is a linear graph whose underlying space is a wedge of $n + 1$ circles. Given H , we choose a path-connected covering space $p : E \rightarrow X$ such that $p_*(\pi_1(E, e_0)) = H$. Now the lifting correspondence.

$$\Phi : \pi_1(X, x_0) / H \rightarrow p^{-1}(x_0)$$

is a bijection. Therefore, E is a k -fold covering of X .

The space E is also a linear graph. Given an edge A of X , the path components of $p^{-1}(A)$ are edges of E , and each is mapped by p homeomorphically onto A . Thus E has k times as many edges as X , and k times as many vertices. It follows that $\chi(E) = k \chi(X)$. Since the fundamental group of X has $n + 1$ free generators, the preceding lemma tells us that $\chi(X) = -n$. Then the number of free generators of the fundamental group of E , which is isomorphic to H , is

$$1 - \chi(E) = 1 - k \chi(X) = 1 + kn.$$

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