Some Exact Solutions of Einstein's Field Equation's for Acceleration Free Imperfect Fluid Source

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Abstract – The research paper provides solution of some exact solution to the Einstein field equation for acceleration free imperfect fluid source with shear viscosity under different conditions. Some physical restriction on the solution have been also discussed.

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INTRODUCTION

Deng and Mannheim [4] have considered an imperfect fluid with shear viscosity, bulk viscosity and heat conduction, taking into consideration the equation of state of the fluid, and allow for the possibility that at earlier time the fluid need not have been commoving with the geometry. To treat this problem in all generality is far too complex and no exact solutions are known. However, some partial studies have been made in the literature which identify some particular exact solutions to the Einstein's equations in a few restricted cases [1,2,3,6,8].

Deng and Mannheim [4] have found some exact solutions to the Einstein's equation with a shear free imperfect fluid source and also with acceleration free ones with shear viscosity [5]. Their solution approaches a locally flat Robertson-Walker one in the large t limits and thus serves as a viable candidate for a realistic cosmological model. This model built of this solutions is found to be free of horizon, entropy and flatness problems while cosmological models built of the solutions [5] are found to have increasing entropy perbaryon and not possess any flatness problem. These solutions also satisfy dominant entropy condition [7].

In this paper we have obtained some exact solutions to the Einstein's equation with an acceleratin free imperfect-fluid source with shear viscosity under different conditions. Some physical restrictions on the solutions have been discussed.

2. THE FIELD EQUATIONS

The field equations are

$$(2.1)\ t_{ij} = \rho u_i u_j + p h_{ij} - 2\eta\sigma_{ij}$$

where t_{ij} is energy momentum tensor, u_1 is four velocity, η (r,t) is shear viscosity coefficient, ρ(r,t) & p(r,t) are the standard energy density and pressure of the fluid and

$$(2.2)\ h_{ij} = g_{ij} + u_i u_j$$

$$(2.3)\ 2\ \sigma_{IJ} = h_i^{\alpha} h_j^{\beta}\left(u\alpha;\beta + u\beta;\alpha - \tfrac{2}{3} g_{\alpha\beta} u^k_{;k}\right).$$

Wetake the geometry to be spherically symmetric about a single point and thus isotropic but not homogeneous at arbitray times. Further, we take the geometry to be acceleration-free (viz. $u^{\alpha} u_{\beta;\alpha} = 0$), so that the most general admissible metric then takes the form

$$(2.4)\ ds^2 = -dt^2 + e^{2\lambda} + dr^2 + X^2(d\theta^2 + sin^2\theta d\emptyset^2),$$

where λ and X are functions of r and t only; while the fluid four-velocity vector itself then simplifies to

$$(2.5) u_i = (-1,0,0,0)$$

So that the fluid is comoving with the geometry. In this geometry the Einstein equations take the form

$$(2.6) K_p = \frac{1}{X^2} + 2\dot{\lambda}\frac{\dot{X}}{X} + \frac{\dot{X}^2}{X^2} - e^{-2\lambda}\left[2\frac{X''}{X} - 2\lambda'\frac{X'}{X} + \frac{X'^2}{X^2}\right]$$

$$(2.7)\ K\left[p - \frac{4}{3}\eta\left[\dot{\lambda} - \frac{\dot{X}}{X}\right]\right] = -\frac{1}{X^2} - 2\frac{\ddot{X}}{X} - \frac{\dot{X}^2}{X^2} + e^{-2\lambda}\frac{X'^2}{X^2}$$

$$(2.8)\ K\left[p + \frac{2}{3}\eta\left[\dot{\lambda} - \frac{\dot{X}}{X}\right]\right] = -\ddot{\lambda} - \dot{\lambda}^2 - \dot{\lambda}\frac{\dot{X}}{X} - \frac{\ddot{X}}{X} + e^{-2\lambda}\left[\frac{X''}{X} - \frac{\lambda' X'}{X}\right]$$

$$(2.9)\ \frac{\dot{X}'}{X} - \frac{\dot{\lambda}X'}{X} = 0$$

Where K denotes the quantity 8^{π}G, so that the Bianchi identities impose the following two constraints on the fluid :

$$(2.10)\ \rho + \left[\dot{\lambda} + 2\frac{\dot{X}}{X}\right](\rho + P) = \frac{4}{3}n\left[\dot{\lambda} - \frac{\dot{X}}{X}\right]^2$$

And

$$(2.11)\ P' = \frac{3}{4}\frac{\partial}{\partial r}\left[\eta\left\{\dot{\lambda} + \frac{\dot{X}}{X}\right\}\right] + 4\eta\left[\dot{\lambda} - \frac{\dot{X}}{X}\right]\frac{X'}{X}$$

(In passing we note that because of the conservation of the energy-momentum tensor we find that unlike the perfect fluid case where some acceleration is necessary to support a pressure gradient, for an imperfect fluid this gradient may be supported by the viscosity instead)

3. SOLUTION OF THE FIELD EQUATIONS

Equations (2.9) on integration gives

$$(3.1) e^{\lambda} = \left|\frac{X'}{\emptyset(r)}\right|,$$

Where $\emptyset$(r) is an integration function.

Now following are the general approach as given by Deng and Mannheim [4] we impose the physically motivated boundary condition that the metric asymptotically approach a Robertson-Walker one. Taking this asymptotic metric to be of the form

$$(3.2)\ X = re^{\lambda}, e^{\lambda} = \frac{K(t)}{\left\{K_1 + \frac{K_2 r^2}{4}\right\}}\ (K = -1,0,1)$$

In isotropic co-ordinate system then requires the function $\emptyset$(r) to be of the form

$$(3.3) \emptyset_k(r) = \frac{K_1 - K_2\frac{r^2}{4}}{K_1 + K_2\frac{r^2}{4}} \text{ or } \frac{1 - \frac{K_2 r^4}{4}}{1 + \frac{K_2 r^2}{4}}\ (\text{when } K_1 = 1)$$

at all times. Further, eliminating λ from equations (2.6) – (2.8) and (3.1), we get

$$(3.4)\ K_\rho = \frac{1}{X^2 X}\frac{\partial}{\partial r}[X(1 - \emptyset^2) + X^2 X]$$

$$(3.5) K\left[\rho - \frac{4}{3}\eta\left\{\frac{\dot{X}}{X'} - \frac{\dot{X}}{X}\right\}\right] = -\frac{1}{X^2}[1 - \emptyset^2 + 2\ddot{X}X + \dot{X}^2],$$

$$(3.6)\ K\left[\rho + \frac{2}{3}\eta\left\{\frac{\dot{X}'}{X} + \frac{\dot{X}}{X}\right\}\right] = -\frac{1}{2XX}\frac{\partial}{\partial X}[1 - \emptyset^2 + 2\ddot{X}X + \dot{X}^2],$$

To solve these equations, we choose an equation of state of the form

$$(3.7) \rho(\text{r,t}) = ap\ (\text{r, t})$$

(For early radiation dominated universe $a = 3$) and for matter dominated universe, we have

$$(3.8)\ \ \text{p (r, t)} = 0$$

Because of equation (3.7) for early radiation dominated universe and, because of our simplifying choice of vanishing bulk viscosity coefficient, the fluid energy-momentum tensor is traceless. This tracelessness condition then leads to

$$(3.9)\ \frac{\partial}{\partial r}\left[X(1 - \emptyset^2) + X\frac{\partial}{\partial t}(X\dot{X})\right] = 0$$

Which on integration yields

$$(3.10)\ X\left[1 - \emptyset^2 + \frac{\partial}{\partial t}(X\dot{X})\right] = \Psi(\text{t})$$

Where $\Psi(t)$ is an arbitrary function of t.

In general it is quite difficult to find exact solutions for (3.10). However, we find some solutions in certain simplified cased :

Case – I when $\Psi(t) = 0$

(Inhomogeneous Solutions When $\Psi(t) = 0$)

In this case we consider equation (3.10) and study some possible cases when $\Psi(t) = 0$. Here we have to obtain exact solutions for each of the three special geometries ($k_2 = -1,0,1$) associated with Eq. (3.3) and which do uniformly approach the Robertson- Walker one with there being no spatially singular point at all except at very early times. To obtain these solutions we note first that with the use of equation (3.3) equation (3.10) may now be written as

$$(3.11)\ \frac{k_1 k_2 r^2}{\left(k_1+\frac{k_2 r^2}{4}\right)^2}+\frac{\partial}{\partial t}(X\dot{X})=0$$

When $\Psi(t)$ is zero. The most general solution to Eqn. (3.11) is

$$(3.12)\ X=\frac{r}{k_1+\frac{k_2 r^2}{4}}\left[-k_1 k_2 t^2+\alpha(r)t+\beta(r)\right]^{\frac{1}{2}}$$

$$\text{Or, } X=\frac{r}{1+\frac{k_2 r^2}{4}}\left[-k_2 t^2+\alpha(r)t+\beta(r)\right]^{\frac{1}{2}} \qquad (\text{when } k_1=1)$$

Where α and β are functions of r. If α and β were both independent of r, the solution would simply be the Robertson-Walker one at all times. We investigate the cases in which at least one of them is not constant for different values of k_2.

Case – 1A (Zero-Curvature case) i.e. k_2 = 0.

Here in this case it is easy to see that $\phi_0(r)$ = 1.

Now $\alpha(r)$ must be a constant for X to approach the Robertson-Walker from asymptotically. Thus we have in general

$$(3.13)\ X=(r,t)=r\left[G\left(t-L(r)\right)\right]^{\frac{1}{2}}$$

Where G is a constant.

From Eqs. (3.1) and (3.4) – (3.6) we obtain

$$(3.14)\quad e^{\lambda}=\frac{G^{\frac{1}{2}}}{\sqrt{t-L(r)}}\left[t-L(r)-\frac{1}{2}rL'(r)\right], t-L(r)+\frac{rL'(r)}{6}$$

$$(3.15)\ K_\rho=\frac{3}{4}\frac{1}{[t-L(r)]^2\ t-L(r)-\frac{rL'(r)}{2}}$$

$$(3.16)\ \eta=\frac{1}{4k}\frac{1}{\{t-L(r)\}}$$

Equation (3.15) should that the energy density has a double pole at t = L(r), a single pole at $t=L(r)+rL'\frac{(r)}{2}$ and also a zero at $t=L(r)-rL'\frac{(r)}{6}$. To have a physical model we need to choose the pole at the late r time to be the big bang singularity so that there is then no singularity after the big bang. If $rL'(r)$ is less then zero the big bang should occur at time $t=L(r)$. But the numerator should then be negative at a t near L(r). This case is thus ruled out. If we set

$$(3.17) rL'(r)\geq 0$$

The big bang occurs at

$$(3.18)\ t=t_0(r)\equiv L(r)+\frac{1}{2}rL'(r)\geq L(r).$$

With equations (3.17) and (3.18) it can be shown that the dominant energy condition is satisfied. Also here we have an inhomogeneous big bang, though no specification of L(r) has been given so far other than Eq. (3.17) which only requires that L(r) be monotonically increasing. Thus practically one can choose almost any function L(r). Below we consider some physically motivatical forms for the function L(r) (i) we may choose L such that the big bang takes place uniformly, i.e., so that $t_0(r)=0.$ Then we take

$$(3.19)\quad L(r)+\frac{1}{2}rL'(r)=0$$

So that we obtain

$$(3.20)\qquad L(r)=\frac{-\alpha}{r^2},$$

Where α has to be positive according to Eq. (3.17) which then makes n nicely positive according to Eq. (3.16). With the use of Eq. (3.20), Eqs. (3.13) – (3.15) then reduce to

$$(3.21)\ x(r,t)=r\left[G\left(t+\frac{\alpha}{r^2}\right)\right]^{\frac{1}{2}},$$

$$(3.22)\qquad e^{\lambda}=\frac{G^{\frac{1}{2}}.t}{\left(t+\frac{\alpha}{r^2}\right)^{\frac{1}{2}}},$$

$$(3.23)\qquad K_\rho=\frac{3}{4t}\frac{\left(t+\frac{4\alpha}{3r^2}\right)}{\left(t+\frac{\alpha}{r^2}\right)^2}.$$

The big bang thus takes place uniformly at t = 0 as required. However, we note that the metric is also singular at r = 0 where it never approaches the Robertson-Walker form. (ii) We may also choose L(r) that is bounded and nowhere singular throughout the entire three-space. The metric would then uniformly approach the Robertson-Walker one t $\rightarrow\infty$. But the big bang would now have to occur non-uniformly. Accordingly we would have a big bang which is space dependent. After enough time has elapsed to allow the big bang to take place at every possible special point, the metric would then become non-singular everywhere. A simple example of such L(r) is

$$(3.24) \quad L(r) = \frac{\alpha r^2}{r^2+r_0^2}$$

Which satisfies Eq. (3.17) provided the constant α is positive. With the use of (3.24) we find

$$(3.25) \quad x = G^{\frac{1}{2}} r \left[t - \frac{\alpha r^2}{r^2+r_0^2}\right]^{\frac{1}{2}},$$

$$(3.26) \quad e^{\lambda} = \frac{G^{\frac{1}{2}}}{\left[t - \frac{\alpha r^2}{(r^2+r_0^2)}\right]^{\frac{1}{2}}} \left[t - \frac{\alpha r^2(r^2+2r_0^2)}{(r^2+r_0^2)^2}\right],$$

$$(3.27) \quad K_{\rho} = \frac{3}{4} \frac{1}{\left[t - \frac{\alpha r^2}{(r^2+r_0^2)}\right]^2} \frac{\left[t - \frac{\alpha r^2\left(r^2+\frac{2}{3}r_0^2\right)}{(r^2+r_0^2)^2}\right]}{\left[t - \frac{\alpha r^2(r^2+2r_0^2)}{(r^2+r_0^2)^2}\right]}$$

The big bang thus takes place at

$$(3.28) \quad t = t_0(r) \equiv \frac{\alpha r^2(r^2+2r_0^2)}{(r^2+r_0^2)^2}$$

With both ρ and η being positive at all times $t > t_0(r)$ as required.

Similarly we can discuss the cases where $k_2 = 1$ and $k_2 = -1$

Case – II when $\Psi(t) \neq 0$

Here we consider the case in which our metric is to asymptotically approach a flat $(k_2 = 0)$ Robertson-Walker metric. Clearly then $\phi(r) = 1$ and equation (3.10) goes to the form

$$(3.29) \quad X \frac{\partial}{\partial t}(X\dot{X}) = \Psi(t)$$

This equation can be solved for different choices of $\Psi(t)$. We take the solution of the form

$$(3.30) \quad X = [\alpha_1(r)\beta_1(t) + \alpha_2(r)\beta_2(t)]^{\frac{2}{3}}$$

Where α_1 and α_2 are not proportional to each other and nor β_1 are β_2 and Inserting equation (3.12) into (3.11) gives an equation that can be satisfied in a finite number of inequivalent ways with the function $\Psi(t)$ being restricted to take only certain specific forms with our additional asymptotic requirement that the metric becomes Robertson-Walker at large time. We can obtain four different solutions for the function X (r,t) which are of interest to cosmology. Substituting four respective values ofX (r,t) in equations (3.1) and (3.4)- (3.6) we can get the values of e^{λ}, ρ and η in each case.

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