

Theory of Some Sequence Spaces and Matrix Transformations: A Review

Sonam Ahuja*

Professor, Department of Mathematics, K. L. Mehta Dayanand College, Faridabad

Abstract – In most cases the most general linear operator from one sequence space into another is actually given by an infinite matrix and therefore the theory of matrix transformations has always been of great interest in the study of sequence spaces. The study of general theory of matrix transformations was motivated by the special results in summability theory. This paper is a review article which gives almost all known results on matrix transformations.

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INTRODUCTION

The study of convergence of series is an art. The work done prior to the time of Leonard Euler (1707-1783), was concerned essentially with orthodox examination of convergence and those series which did not converge were of little or no importance. Augustin-Louis Cauchy (1789-1875) and Neil H. Abel (1802-1829) were among the pioneers in introducing a rigorous foundation for

the algebra of infinite series. In his famous treatise; "Analyse Algebriques", Cauchy (1821) gave a rigorous definitive formulation of the new classical concept of "convergence" for infinite series. Given an infinite series $\sum a_n$, (x_n) denotes the sequence of its partial sums, i.e.,

$$x_n = \sum_{k=0}^n a_k, \quad (n=0, 1, 2, \dots)$$

Suppose that there is a number l such that (x_n) converges to l , i.e., given $\epsilon > 0$, there exists an integer N such that $|x_n - l| < \epsilon$ for all $n > N$. Cauchy defined the 'sum' of $\sum a_n$ as the number l . Since there cannot exist two such numbers J , this definition of the 'sum' of an infinite series is unique. A series which has a sum in Cauchy's sense is said to be convergent. A series which is not convergent is called divergent.

So greatly emphasized by the idea of this now too familiar concept of convergence that infinite series which diverged were usually dropped as being without any mathematical significance. Indeed, the so-called 'divergent series' were a sort of taboo in the eyes of many mathematicians including even a genius like Abel, who is known to have remarked: "Divergent series are the inventions of the devil and

it is shameful to base on them any demonstration what so ever".

For generalizing the notion of convergence, one can, for example, replace the sequence (x_n) of partial sums of any given series $\sum a_n$ by the sequence of 'Arithmetic Means' or $(C, 1)$ - means of (x_n) , denoted by σ_n where,

$$\sigma_n = \frac{x_0 + x_1 + \dots + x_n}{n+1}.$$

If $\lim_{n \rightarrow \infty} \sigma_n = l$, we call l the '(C,l) - sum' of $\sum a_n$ (or the (C,l) -limit of (x_n)) and say that the sequence (x_n) , or the series $\sum a_n$, is (C,l) - convergent (or (C, 1) - summable) to l . We observe that (i) $\sum a_n$ may be summable (C, 1) to l whenever $\sum a_n$ or (x_n) converges to l and (ii) $\sum a_n$ may be summable (C,l) even in certain cases in which $\sum a_n$ fails to converge, e.g; $1-1 + 1-1+\dots$, although divergent, is summable (C,l) to $1/2$.

More generally, we can replace (σ_n) by the sequence (y_n) defined by the sequence-to-sequence matrix transformation (or A - mean):

$$y_n = \sum_k a_{n,k} x_k \quad (n=0, 1, 2, \dots),$$

the series on the right defining y_n being assumed to converge to y for each n , where $a_{n,k}$ is the element of matrix $A=(a_{n,k})$ in the n th row and k th column. The sequence (x_k) , or the series $\sum a_n$ will be said to be summable A to the sum l , if $\lim_{n \rightarrow \infty} y_n = l$.

We observe that the interest in general matrix transformation theory was, to some extent, stimulated by special results in summability theory, which were obtained by Cesaro (1890), Borel (1921) and others. It was however the celebrated German mathematician Toeplitz (1881-1940) who, in 1911, brought the methods of linear space theory to bear on problems connected with matrix transformations on sequence spaces. Toeplitz characterized all those infinite matrices $A=(a_{n,k})$ which map the space c into itself, leaving the limit of each convergent sequence invariant. To be explicit, he gave the necessary and sufficient conditions on A for $y_n \rightarrow l$ (as $n \rightarrow \infty$), whenever $x_k \rightarrow l$ (as $k \rightarrow \infty$).

CESARO SEQUENCE SPACES

In 1970, Shiue studied and discussed the Cesaro sequence spaces $^{ces}_p$ and $^{ces}_\infty$ which were defined in Problem 2 in New Archives for Mathematics, 3, XVII, No. 1 (1968) as:

$$^{ces}_p := \{x \in \omega : \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^n |x_k|^p < \infty\}, 1 < p < \infty;$$

$$^{ces}_\infty := \{x \in \omega : \sup_m \frac{1}{m} \sum_{k=1}^m |x_k| < \infty\}, p = \infty.$$

with the finite norms :

$$\|x\|_p = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n |x_k|^p \right)^{1/p} \right)^p, 1 \leq p < \infty;$$

and

$$\|x\|_\infty = \sup_n \frac{1}{n} \sum_{k=1}^n |x_k|,$$

respectively.

In 1977, Lim defined the space $ces(p)$ for $p = (p_r)$, with

$\inf p_r > 0$ as :

$$ces(p) := \{x=(x_k) : \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x_k|^{p_r} \right) < \infty\},$$

where \sum_r denotes a sum over the ranges $2^r \leq k < 2^{r+1}$, which is paranormed by

$$g(x) = \left(\sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x_k|^{p_r} \right)^{1/M} \right)^M,$$

where $M = \max(1, \sup_r p_r)$, with $\sup_r p_r < \infty$.

In 1977, Johnson and Mohapatra (1979) generalized the space $^{ces}(p)$ to $ces(p, q)$ defined by

$$ces(p, q) := \{x \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k|^{p_r} \right) < \infty\},$$

Where $p = (p_r)$ is positive sequence of real numbers and $q=(q_r)$ is such that

$$Q_{2^r} = q_{2^r} + q_{2^r+1} + \dots + q_{2^r+1-1}$$

If $q_n=1$, for all n , then $ces(p, q)$ reduces to $ces(p_n)$ studied by Lim. Also if $p_n=p$ and $q_n=1$ for all n , then $ces(p, q)$ reduces to $cesp$ defined and discussed by Shiue (1970).

DIFFERENCE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

An Orlicz function is a function $M:[0, \infty) \rightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x)+M(y)$, then this function is called modulus function, defined and discussed by Ruckle and Maddox .

An Orlicz function M is said to satisfy Δ_2 condition for all values of u , if there exists a constant $K>0$, such that $M(2u) \leq KM(u)$, ($u \geq 0$). The Δ_2 condition is equivalent to $M(lu) \leq Kl M(u)$ for all values of u and for $l > 1$.

An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x q(t)dt,$$

where $q(t)$, known as kernel of M , is right differentiable for $t \geq 0, q(0) = 0, q(t) > 0$, for $t > 0$; $q(t)$ is non-decreasing and $q(t) \rightarrow \infty$, as $t \rightarrow \infty$.

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz functions to construct an Orlicz sequence space ℓ^M defined by $\ell^M := \{x \in \omega : \sum_{k=1}^{\infty} M \left(\frac{|x_k|}{\rho} \right) < \infty\}$, for some $\rho > 0$,

The space ℓ^M with the norm :

$$\|x\| = \inf \{ \rho > 0 : \sum_{k=1}^{\infty} M \left(\frac{|x_k|}{\rho} \right) \leq 1 \},$$

is a Banach space.

We use throughout $p = (p_r)$ a sequence of positive real numbers

$$E, F = c_0, c \text{ or } \ell_\infty,$$

$$\Delta x_k = x_k - x_{k+1},$$

$$\Delta^2 x_k = \Delta(\Delta x_k) = \Delta x_k - \Delta x_{k+1},$$

$$\lim_m \sum_k \left| \sum_{n=0}^{\infty} a_{nk} \right| = 0 \tag{3}$$

holds; if, and only if, the condition

$$\sum_k \left| \sum_{n=0}^m a_{nk} \right| \tag{4}$$

Converges uniformly in m, and

$$\sum_n a_{nk} \tag{5}$$

converges for all k, hold.

Theorem C. $A \in (\ell_\infty, \gamma_0)$ if, and only if, the condition

$$\lim_m \sum_k \left| \sum_{n=0}^m a_{nk} \right| = 0, \tag{6}$$

holds.

Theorem D. $A \in (c, \gamma)$ if, and only if, the conditions (1),

(5) and

$$\sum_n \sum_k a_{nk} \tag{7}$$

converges, hold.

$A \in (c, \gamma)$ with $\lim_m (t(Ax))_n = \lim_n x_n$, for all $x \in c$,

if, and only if, the conditions (1),

$$\sum_n a_{nk} = 0 \tag{8}$$

For all k and

$$\sum_n \sum_k a_{nk} = 1, \tag{9}$$

hold.

Theorem E. $A \in (\ell, \gamma)$ if, and only if, the conditions (5)

and

$$\sup_{m,k} \left| \sum_{n=0}^m a_{nk} \right| < \infty, \tag{10}$$

Nanda characterized the matrices $(\ell_\infty(p), \gamma)$, $(c(p), \gamma)$ and $(\ell(p), \gamma)$ which are the generalizations of

H. Kizmaz (1981) has introduced and studied the difference sequence spaces

$$E(\Delta) := \{x \in \omega : \Delta x \in E\}.$$

Recently, Mikail Et has defined the spaces

$$E(\Delta^2) := \{x \in \omega : \Delta^2 x \in E\}.$$

Qamaruddin (2006) has studied the following difference sequence spaces of Orlicz type with the aid of certain convex function M,

$$E(\Delta, M) := \{x = (x_k) : [M \left\{ \frac{|\Delta x_k|}{\rho} \right\}] \in E, \text{ for some } \rho > 0\},$$

$$E(\Delta, M, p) := \{x = (x_k) : [M \left\{ \frac{|\Delta x_k|}{\rho} \right\}] \in E(p) \text{ for some } \rho > 0\}$$

We define the new difference Orlicz space in the same manner as Mikail Et

$$E(\Delta^2, M) := \{x = (x_k) : [M \left\{ \frac{|\Delta^2 x_k|}{\rho} \right\}] \in E, \text{ for some } \rho > 0\},$$

$$E(\Delta^2, M, p) := \{x = (x_k) : [M \left\{ \frac{|\Delta^2 x_k|}{\rho} \right\}] \in E(p), \text{ for some } \rho > 0\},$$

With the help of sequence of Orlicz functions $M = (M_k)$, we also define

$$F(\Delta^2, M) := \{x = (x_k) : [M_k \left\{ \frac{|\Delta^2 x_k|}{\rho} \right\}] \in F, \text{ for some } \rho > 0\}$$

A CLASS OF MATRIX TRANSFORMATIONS

We know the following results.

Theorem A. $A \in (\ell_\infty, \gamma_\omega) = (c, \gamma_\omega) = (c_0, \gamma_\omega)$ if, and only if, the

condition

$$\sup_m \sum_k \left| \sum_{n=0}^m a_{nk} \right| < \infty \text{ holds:} \tag{1}$$

Theorem B. $A \in (\ell_\infty, \gamma)$ if, and only if, the condition :

$$\lim_m \sum_k \left| \sum_{n=0}^m a_{nk} \right| = \sum_k \left| \sum_n a_{nk} \right| \tag{2}$$

holds; if, and only if, the condition

Theorems B, D and E mentioned above respectively.

Theorem F. $A \in (c(p), \gamma)$ if, and only if, the conditions:

$$D = \sup_n \sum_k |b_{nk}| B^{-1/p_k} < \infty, \tag{11}$$

For some integer $B > 1$,

there exists $\alpha_k \in \mathbb{C}$ such that

$$\lim_{n \rightarrow \infty} b_{nk} = \alpha_k \text{ (for all } k), \tag{12}$$

and there exists $\alpha \in \mathbb{C}$ such that

$$\lim_{n \rightarrow \infty} \sum b_{nk} = \alpha, \tag{13}$$

hold

Theorem G. $A \in (\ell_\infty(p), \gamma)$ if, and only if, the conditions and for all integer $N > 1$,

$$\sum_k |b_{nk}| N^{1/p_k} \tag{14}$$

Converges uniformly in n, hold.

Theorem H. $A \in (\ell(p), \gamma)$ if, and only if, the conditions (12) and (15) for all integers $n, m \geq 1, 1 \leq s \leq \infty$, there exists an integer $B > 1$, such that

$$\begin{aligned} C(B) &= \sup_n \sum_{k=m}^s |a_{nk}|^{p_k} B^{-p_k} > \infty \text{ (} 1 < p_k < \infty), \\ &= \sup_{n,k} |b_{nk}|^{p_k} > \infty \text{ (} 0 < p_k \leq 1), \end{aligned} \tag{15}$$

hold.

Our aim here is to characterize the matrix classes $(M_\infty(p), \lambda)$, where λ is one of the sequence spaces γ_∞, γ and γ_0 . We also investigate some matrix transformations in the case $p \in \mathbb{Q}$.

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Corresponding Author

Sonam Ahuja*

Faculty, Department of Commerce, PT. JLN Govt.
College, Faridabad