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Applications of Singularity Perturbed Differential Equations: Some Numerical Treatment

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Abstract – Singular perturbation problems (SPPs) arise frequently in fluid dynamics, quantum mechanics, chemical reactor theory and several other branches of applied mathematics. A wide variety of methods such as perturbation methods, Pad6 approximation methods and numerical methods are available in the literature to solve these problems. The numerical treatment of SPPs has received a significant amount of attention in recent years. It is a well-known fact that the solutions of SPPs exhibit a multiscale character.

In this talk, I will discuss the role of numerical analysis in the design of numerical al¬gorithms to approximately solve certain classes of singularly perturbed differential equations. The solutions of singularly perturbed differential equation have narrow layer regions in the domain, where the solution exhibits steep gradients. Classical numerical methods suffer major defects in these regions. Alternative computational approaches will be discussed and the central issues in the associated numerical analy¬sis of these layer-adapted algorithms will be outlined.

We present new results in the numerical analysis of singularly perturbed convection-diffusion- reaction problems that have appeared in the last five years. Mainly discussing layer-adapted meshes, we present also a survey on stabilization methods, adaptive methods, and on systems of singularly perturbed equations.

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INTRODUCTION

In the present study, some niimerical methods are suggested to solve singularly perturbed two-point boundary-value problems (BVPs) for second-order ordinary differential equations (ODEs) and initial-boundary value problems for parabolic partial differential equations (PDEs). Wherever possible error estimates for these methods are derived. An extensive amount of literature on the numerical methods for singular perturbation problems (SPPs) is available now-a-days. Some references and a detailed summary of some of these methods relevant to the thesis are presented here.

The niunerical treatment of SPPs has received a great deal of attention in the past few decades. *1%* is well-known that solutions of SPPs have multiscale character. The presence of small parameter(s) in these problems prevent us from obtaining satisfactory numerical and asymptotic solutions by direct or classical methods. The solutions of these problems typically contain layers known as "boundary or interior layers". For numerical solutions, various finite difference schemes have been proposed in the literature in order to guarantee stability of the schemes for all values of the parameter *£*. Careful examination of numerical results from such schemes on uniform grids shows that for fixed (small) values of the perturbation parameter, the maximum pointwise error usually increases as the mesh is refined - because of the presence of "boundary or interior layers" - until the mesh diameter is comparable in size to the parameter. This behaviour is clearly unsatisfactory. Therefore a separate treatment is necessary to deal with such problems.

A singular perturbation problem is a problem which depends on a parameter (or parameters) in such a uay that solutions of the problem behave nonuniformly as the parameter tends toward some limiting value of interest. Such singular perturbation problems involving differential equations arise in many areas of interest, e.g. modelling of semi-conductor devices, aerodynamics, fluid mechanics, thin shells. We illustrate some of the nonuniformities that occur with some simple prototypes.

There are two main approaches to solving differential equations numerically:

(1) Finite Difference Methods

In one dimension, divide the interval [a,b] into N sub-intervals

$$a = x_0 < x_1 < \cdots\cdots < x_N = b$$

Replace y and its derivatives in the differential equation by suitable (difference) approximations e.g. replace

$$y'(x_j) \text{ by } (u_{j+1} - u_j)/(x_{j+1} - x_j)$$

and then replace the coefficients of the derivatives by an appropriate approximation.

e.g. on $[x_j, x_{j+1}]$ replace $a(x)$ by $a(x_j)$ or $a(x_{j+1})$

A system of algebraic equations is then solv/ed to generate a set of points $\{u_j\}$ as an approximation to the set $\{y(x_j)\}$

(2) Finite Element Methods

A function u(x) is generated by discretizing a weak form of the differential equation. This function approximates the solution y(x) globally.

In this note we will confine the discussion to finite difference methods.

Classical numerical methods perform badly (to say the least) when applied to singularly perturbed problems. In particular, their atrocious behaviour is most noticeable in non-self-adjoint problems.

Singularly perturbed differential equations arise in many branches of science and engineering. The solutions of such equations have boundary and interior layers. That is, there are thin layer(s) where the solution changes rapidly, while away from the layer(s) the solution behaves regularly and changes slowly. So the numerical treatment of singularly perturbed differential equations gives major computational difficulties, and in recent years, a large number of special purpose methods have been developed to provide accurate numerical solutions which cover mostly second order equations. But only a very few authors have developed numerical methods for singularly perturbed higher order differential equations [IT]. Moreover, most of them have concentrated only on the problems with smooth data. Of course, some authors have recently considered Singular Perturbation Problems (SPPs) for second order ODEs with discontinuous source term and discontinuous convection coefficient. Due to the discontinuity at one or more points in the interior domain, this gives rise to an interior layer(s) in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Therefore, these types of SPPs have to be dealt with separately and carefully. In this paper, an asymptotic numerical method for singularly perturbed react ion-diffusion type third order ODE with a discontinuous source term is developed. The classification of singularly perturbed higher order problems (reaction-diffusion/convection-diffusion) depend on how the order of the original equation is affected if one sets *e* = 0. If the order is reduced by one, we say that the problem is of convection-diffusion type, and of reaction-diffusion type if the order is reduced by two.

Systems involving several time scales often assume the prototypical form

$$\dot{x}^\epsilon = f(x^\epsilon, y^\epsilon, \epsilon)$$

$$\dot{y}^\epsilon = \frac{1}{\epsilon} g(x^\epsilon, y^\epsilon, \epsilon)\,, \quad x^\epsilon(0) = x_0,\, y^\epsilon(0) = y_0 \qquad (1)$$

where $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$ and $0 < \epsilon \ll 1$ is a small parameter and we use the "overdot" to denote derivatives with respect to the free variable, i.e., $\dot{z}(t) = dz/dt$. If $f(\cdot,\cdot,\epsilon)$ and $g(\cdot,\cdot,\epsilon)$ are globally Lipschitz and uniformly bounded in ϵ, then $\dot{y}^\epsilon(t)$ will be of order $1/\epsilon$ faster than $\dot{x}^\epsilon(t)$. Accordingly, we call *x* the slow variables and *y* the fast variables of the system.

The analyses of singularly perturbed differential equations such as (1) often boil down to linear operator equations of the type (see, e.g., for a treatment of stochastic systems)

$$\partial\phi^\epsilon(u,t) = \mathcal{L}^\epsilon\phi^\epsilon(u,t)\,, \quad \phi^\epsilon(u,0) = \psi(u)\,. \qquad (2)$$

Here $\mathcal{L}^\epsilon$ is a differential operator that is defined on some Banach space subject to suitable boundary conditions and $u \in \mathbf{R}^{n+m}$ is a shorthand for *(x, y).* If we confine our attention to the aforementioned class of problems (averaging or geometric singular perturbation) the operator typically takes the form

$$\mathcal{L}^\epsilon = \mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_1$$

Where $\mathcal{L}_0$ and $\mathcal{L}_1$ "generate" the slow and fast dynamics, respectively. (The properties of $\mathcal{L}_1$ depend on the actual problem and will be discussed later on in the text.) Notice that $\mathcal{L}_0$ and $\mathcal{L}_1$ may still depend on ϵ, but the dominant singularity is as sketched. We seek a perturbative expansion of the solution of (2) that has the form

$$\phi^\epsilon = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots\,.$$

Hence (1.6) can be recast as

$$\frac{1}{\epsilon}\mathcal{L}_1\phi_0 + (\mathcal{L}_1\phi_1 + \mathcal{L}_0\phi_0 - \partial\phi_0) = \mathcal{O}(\epsilon)\,,$$

and equating powers of ϵ yields a hierarchy of equations the first two of which are

$$\mathcal{L}_1\phi_0 = 0$$

$$\mathcal{L}_1\phi_1 = \partial\phi_0 - \mathcal{L}_0\phi_0\,.$$

The first equation implies that the lowest-order perturbation approximation of (1) lies in the nullspace of $\mathcal{L}_1$, which typically entails a condition of the form $\phi_0 = \phi_0(x)$. Averaging or geometric singular perturbation theory now consists in finding an appropriate closure of the second equation subject to $\phi_0 \in \ker \mathcal{L}_1$. This results in an effective equation for $\phi \approx \phi_0$, namely,

$$\partial\phi(x,t) = \bar{\mathcal{L}}\phi(x,t)$$

Once the effective linear operator $\bar{\mathcal{L}}$ has been computed from $\mathcal{L}_0$ and $\mathcal{L}_1$, the result can be reinterpreted in terms of the corresponding differential equation to give $\dot{x} = \bar{f}(x)\,,\quad x(0) = x_0$

SINGULAR PERTURBATION ORDINARY DIFFERENTIAL EQUATION : GEOMETRIC THEORY

Consider the following system of singularly perturbed ordinary differential equation

$$\dot{x} = f(x,y)\,,\quad x(0) = x_0 \quad (3)$$

$$\epsilon\dot{y} = g(x,y)\,,\quad y(0) = y_0$$

for $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$ and $0 < \epsilon \ll 1$. Here and in the following we omit the parameter ϵ, i.e., we write $x = x^\epsilon$, $y = y^\epsilon$ and so on; the meaning should be always clear from the context and we indicate otherwise when not. We let $\varphi^t_\xi : \mathbf{R}^m \to \mathbf{R}^m$, $y(t) = \varphi^t_\xi(y_0)$ denote the solution of the associated system

$$\dot{y} = g(\xi,y)\,,\quad y(0) = y_0 \quad (4)$$

and recall that the first (i.e., slow) equation in (3) can be approximately viewed as an equation of the form

$$\dot{x} = f(x, \varphi_x^{t/\epsilon}(y_0))\,,\quad x(0) = x_0$$

Let us suppose that

$$\lim_{t\to\infty} \varphi^t_\xi(y_0) = m(\xi)$$

exists independently of the initial value $y(0) = y_0$ and uniformly in $x = \xi$, i.e., the rate of convergence is independent of the slow variable. In particular, $\lim_{\epsilon\to 0} \varphi_x^{t/\epsilon}(y_0) = m(x)$ for fixed and, by the above argument, we may replace (4.1) by the equation

$$\dot{x} = f(x, m(x))\,,\quad x(0) = x_0$$

whenever e is sufficiently small. We shall give an example. Example 4.1 Consider the linear system

$$\dot{x} = A_{11}x + A_{12}y\,,\quad x(0) = x_0$$

$$\epsilon\dot{y} = A_{21}x + A_{22}y\,,\quad y(0) = y_0\,.$$

We suppose that $\sigma(A_{22}) \subset \mathbf{C}^-$, i.e., all eigenvalues of A22 lie in the open left complex half-plane. As we shall see this is equivalent to the statement that the fast subsystem is asymptotically stable. The associated system

$$\dot{y} = A_{22}(y + A_{22}^{-1}A_{21}\xi)\,,\quad y(0) = y_0$$

is easily solvable using variation of constants, viz.,

$$\varphi^t_\xi(y_0) = \exp(tA_{22})(y + A_{22}^{-1}A_{21}\xi) - A_{22}^{-1}A_{21}\xi\,.$$

Note that A_{22} is invertible by the stability assumption above. Furthermore

$$\varphi^t_\xi(y_0) \to -A_{22}^{-1}A_{21}\xi \quad \text{as} \quad t \to \infty$$

The vector field for a planar linear system with $A_{11} = A_{22} = -1$ and $A_{12} = A_{21} = -1/2$. The leftmost plot shows various solutions for e whereas the right figure depicts solutions (blue curves) for $\epsilon = 0.05$. It turns out that the solutions quickly converge to the nullcline $\dot{y} = 0$ (fast dynamics) before converging to the asymptotically stable fixed point $(x,y) = (0,0)$ along the nullcline (slow dynamics). Note that the nullcline corresponds to the invariant subspace that is defined by the equation $A_{21}x + A_{22}y = 0$. If $A_{22} < 0$, the subspace is attractive (i.e., asymptotically stable).

SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS: ROBUST NUMERICAL METHODS

In 2009, Linss and Stynes presented a survey on the numerical solution of singularly perturbed systems. In this Section we only comment on some recent results not contained in which sparkle that survey.

First we study systems of reaction-diffusion equations of the form

$$-E^2u'' + Au = f \quad \text{in } (0,1), u(0) = u(1) = 0, \qquad (5)$$

where $E = \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ with $\varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_l$. If the matrix *A* satisfies certain conditions, the asymptotic behavior of such a system is well understood. Assume that *A* has positive diagonal entries, moreover the matrix Γ defined by

$$\gamma_{ii} = 1, \quad \gamma_{ij} = -\left\| \frac{a_{ij}}{a_{ii}} \right\|_\infty \quad \text{for } i \ne j \qquad (6)$$

satisfies $\Gamma^{-1} \ge 0$. Then, in the existence of a solution decomposition is proved. Other authors assume that *A* is an M-matrix or that *A* is point-wise positive definite. See for establishing a connection between positive definiteness and the property $\Gamma^{-1} \ge 0$. In a full asymptotic expansion is derived for positive definite *A* in the case of two equations, including information on analytic regularity.

Systems of convection-diffusion problems are more delicate to handle. Consider first weakly coupled systems of the form

$$Lu := -Eu'' - \text{diag}(b)u' + Au = f, \quad u(0) = u(1) = 0, \qquad (7)$$

assuming $|b_i| \ge \beta_i > 0$ and $\tilde{\Gamma}^{-1} \ge 0$ with

$$\tilde{\gamma}_{ii} = 1, \quad \tilde{\gamma}_{ij} = -\min\left(\left\| \frac{a_{ij}}{a_{ii}} \right\|_\infty, \left\| \frac{a_{ij}}{b_i} \right\|_\infty \right) \quad \text{for } i \ne j. \qquad (8)$$

Then it was shown in for $\nu = 0, 1$

$$\left| u_k^{(\nu)}(x) \right| \le C \begin{cases} 1 + \varepsilon_k^{-\nu} e^{-\beta_k(1-x)/\varepsilon} & \text{if } b_k < 0, \\ 1 + \varepsilon_k^{-\nu} e^{-\beta_k x/\varepsilon} & \text{if } b_k > 0. \end{cases} \qquad (9)$$

When only first order derivatives are considered, there is no strong interaction between the layers of different components unlike the reaction-diffusion case.

But, consider for example a set of two equations with $b_1 > 0$ and $b_2 < 0$ for $\varepsilon_1 = \varepsilon_2$.

Then the layer at *x* = 1 in the first component generates a weak layer at *x* = 1 in the second component; the situation at* = 0 is analogous. Under certain conditions , one can prove the existence of the following solution decomposition for $\nu \le 2$:

$$u_1 = S_1 + E_{10} + E_{11}, \qquad (10)$$

$$u_2 = S_2 + E_{20} + E_{21}$$

$$\left\|S_1^{(\nu)}\right\|_0, \left\|S_2^{(\nu)}\right\|_0 \le C,$$
$$\left|E_{10}^{(\nu)}(x)\right| \le C\varepsilon^{1-\nu}e^{-\alpha(x/\varepsilon)}, \quad \left|E_{11}^{(\nu)}(x)\right| \le C\varepsilon^{-\nu}e^{-\alpha((1-x)/\varepsilon)},$$
$$\left|E_{20}^{(\nu)}(x)\right| \le C\varepsilon^{-\nu}e^{-\alpha(x/\varepsilon)}, \quad \left|E_{21}^{(\nu)}(x)\right| \le C\varepsilon^{1-\nu}e^{-\alpha((1-x)/\varepsilon)}. \qquad (11)$$

Here α is some positive parameter. This observation is important for control problems governed by convection-diffusion equations:

$$\min_{y,q} J(y,q) := \min_{y,q}\left(\frac{1}{2}\|y - y_0\|_0^2 + \frac{\lambda}{2}\|q\|_0^2 \right) \qquad (12)$$

subject to

$$Ly := -\varepsilon y'' + by' + cy = f + q \quad \text{in } (0,1),$$
$$y(0) = y(1) = 0. \qquad (13)$$

For strongly coupled systems of convection-diffusion equations full layer-interaction takes place. Consider the system of two equations

$$Lu := -\varepsilon u'' - Bu' + Au = f, \quad u(0) = u(1) = 0 \qquad (14)$$

assuming

(Vi) *B* is symmetric.

(V_2) $A + 1/2B'$ is positive semidefinite.

(V_3) The eigenvalues of *B* satisfy $|\lambda_{1,2}| > \alpha > 0$ for all x.

If both eigenvalues of *B* are positive, both solution components do have overlapping layers at *x* = 0, the reduced solution solves an initial value problem. But if the eigenvalues do have a different sign, both solution components do have layers at *x* = 0 *and x* = 1; we have full layer interaction. It is remarkable that the reduced solution, in general, does not satisfy any of the given boundary conditions.

Even more complicated are strongly coupled systems with several small parameters of the form

$$Lu := -Eu'' - Bu' + Au = f, \quad u(0) = u(1) = 0. \qquad (15)$$

Some a priori estimates are to find in, information on the layer structure in.

BOUNDARY-VALUE TECHNIQUE FOR SINGULARLY PERTURBED TURNING POINT PROBLEMS

A class of singularly perturbed turning point problems for second-order ordinary differential equations exhibiting twin boundary layers are considered. In order to obtain numerical solution of these problems a computational method in which exponentially fitted difference schemes are combined with classical numerical methods is suggested.

Consider **the** singularly **perturbed TPP:**

$$Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x),\ x \in I = (-1,1),$$

(1a)

$$B_{-1}u(-1) \equiv u(-1) = A,\ B_1u(1) \equiv u(1) = B, \quad (1b)$$

where a(x), b(x) and f(x) are sufficiently smooth functions,

$$a(0) = 0,\ a'(0) < 0. \quad (2a)$$

In order that the solution of (1) satisfies a maximum principle, we require that

$$b(x) \geq 0\ ,\ b(0) > 0. \quad (2b)$$

Also b(x) is required to be bounded below by some positive constant b, so as to exclude the so called resonance cases, that is,

$$b(x) \geq b > 0. \quad (2c)$$

We also impose the following restriction which ensures that there are no other turning points in the interval $\bar{I} = [-1,1]$

$$|a'(x)| \geq |a'(0)/2|,\ x \in \bar{I}. \quad (2d)$$

Under these conditions (2), the TPP (1) has a unique solution having two boundary layers at x = -1 and x = 1,

Let k be such that $k\varepsilon \ll 1$, where k£ be the width or thickness of the boundary layers which are near x =-1 and x =

1. The interval $\bar{I} = [-1,1]$ isnow divided into four subintervals namely,

$I_1 = [-1,-1+k\varepsilon]$, $I_2 = [-1+k\varepsilon, -\delta]$, $I_3 = [\delta, 1-k\varepsilon]$ and $I_4 = [1-k\varepsilon,1]$, where $\delta > 0$ is a small number. Hence, one obtains two types of problems, each containing two differential equation systems, as follows:

1. Inner region problems:

i. The DE (1a) with boundary conditions: $u(-1) = A,\ u(-1+k\varepsilon) = \bar{A};$

ii. The DE (1a) with boundary conditions: $u(1-k\varepsilon) = \bar{B},\ u(1) = B;$

2. Outer region problems:

i. The DE (1a) with boundary conditions: $u(-1+k\varepsilon) = \bar{A},\ u(-\delta) = A^*;$

ii. The DE (1a) with boundary conditions: $u(\delta) = B^*,\ u(1-k\varepsilon) = \bar{B},$

where $\bar{A}, \bar{B}, A^*$ and B^* will be fixed in the next section.

Applying suitable difference schemes to inner and outer region problems, and combining their solutions one gets a numerical solution of the BVP (1) on the interval [-1,1].

TERMINAL BOUNDARY CONDITIONS-

In order to get boundary values $\bar{A}, \bar{B}, A^*$ and B^* consider the asymptotic expansion solution u(x) of (1) given:

$$u(x) = \sum_{i=0}^{n} [u_i(x) + v_i(x) + w_i(x)]\varepsilon^i + O(\varepsilon^{n+1}).$$

In particular, one has

$$|u(x) - [u_0(x) + v_0(x) + w_0(x)]| \leq C\varepsilon, \quad (3)$$

where C is a constant independent of ε, $u_0(x)$ is the solution of

the reduced problem

$$a(x)u_0'(x) - b(x)u_0(x) = f(x),\ x \in I, \quad (4)$$

v_0 and w_0 are given by

$$v_0(x) = [A - u_0(-1)]\exp[-a(-1)(1+x)/\varepsilon], \quad (5)$$

$$w_0(x) = [B - u_0(1)]\exp[a(1)(1-x)/\varepsilon]. \quad (6)$$

As indicated in the last section, values for $\bar{A}, \bar{B}, A^*$ and B^* are respectively taken as $\bar{A} = [u_0(-1+k\varepsilon) + v_0(-1+k\varepsilon) + w_0(-1+k\varepsilon)],$ $\bar{B} = [u_0(1-k\varepsilon) + v_0(1-k\varepsilon) + w_0(1-k\varepsilon)],$ $A^* = [u_0(-\delta) + v_0(-\delta) + w_0(-\delta)]$ and $B^* = [u_0(\delta) + v_0(\delta) + w_0(\delta)].$

INNER REGION PROBLEMS-

The inner region problems for (1) are given by

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x),\ x \in I_1, \quad (7a)$$

$$u(-1) = A,\ u(-1+k\varepsilon) = \bar{A}, \quad (7b)$$

and

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x),\ x \in I_4, \quad (8a)$$

$$u(1-k\varepsilon) = \bar{B},\ u(1) = B. \quad (8b)$$

Because of (2a), $a(x) \geq a > 0$ for $x \in I_1$, and $a(x) \leq -a < 0$, for $x \in I_4$. Therefore the following EFD schemes

are applied to solve numerically (7) and (8) :

$$L_1^h u_i \equiv \varepsilon\sigma_i(\rho)\ D_+D_-u_i + a(x_i)D_0u_i - b(x_i)u_i = f(x_i),\ x_i \in I_1,\quad i = 1(1)N-1, \quad (9a)$$

$$u_0 = A,\ u_N = \bar{A}, \quad (9b)$$

and

$$L_1^h u_i \equiv \varepsilon\sigma_i(\rho)\ D_+D_-u_i + a(x_i)D_0u_i - b(x_i)u_i = f(x_i),\ x_i \in I_4,\quad i = 1(1)N-1, \quad (10a)$$

$$u_0 = \bar{B},\ u_N = B, \quad (10b)$$

Where $\sigma_i(\rho) = [\rho a(x_i)\ \coth(\rho a(x_i)/2)]/2,$

$$D_+D_-u_i = (u_{i+1} - 2u_i + u_{i-1})/h^2,\ D_0u_i = (u_{i+1} - u_{i-1})/2h,\ \rho = h/\varepsilon.$$

OUTER REGION PROBLEMS-

The outer region problems for (1) are given by

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x),\ x \in I_2, \quad (11a)$$

$$u(-1+k\varepsilon) = \bar{A},\ u(-\delta) = A^*, \quad (11b)$$

and

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x),\ x \in I_3, \quad (12a)$$

$$u(\delta) = B^*,\ u(1-k\varepsilon) = \bar{B}, \quad (12b)$$

To solve numerically (11) and (12) the following classical upwind difference schemes are applied:

$$L_2^h u_i \equiv \varepsilon D_+D_-u_i + a(x_i)D_+u_i - b(x_i)u_i = f(x_i),\ x_i \in I_2,\quad i = 1(1)N-1, \quad (13a)$$

$$u_0 = \bar{A},\ u_N = A^*, \quad (13b)$$

and

$$L_3^h u_i \equiv \varepsilon\ D_+D_-u_i + a(x_i)D_-u_i - b(x_i)u_i = f(x_i),\ x_i \in I_3,\quad i = 1(1)N-1 \quad (14a)$$

$$u_0 = \bar{B},\ u_N = B^*, \quad (14b)$$

where

$$D_+u_i = (u_{i+1} - u_i)/h,\ D_-u_i = (u_i - u_{i-1})/h.$$

SOLUTION OF THE ORIGINAL PROBLEM-

After solving both inner and outer region problems, their solutions are combined to obtain an approximate solution to the original problem (1) over the interval [-1,1]. This process is repeated by increasing the value of k (thus widening the inner region), until the solution profiles do not differ materially from iteration to iteration. For computational purposes the following absolute error criteria can be used:

$$|u(x_i)^{m+1} - u(x_i)^m| \leq \eta, \quad (15)$$

Where $u(x_i)^m$ is the rath iteration of inner region solution at the ith mesh point and η is the prescribed tolerance bound.

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