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Some New Results on Frechet Spaces with Nuclear Kothe Quotients

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Abstract* – *Our aim is to construct a separable Frechet non-Banach Space X with a continuous norm to have a quotient Y with a continuous norm and a basis .Here in addition, Y can be chosen to be nuclear, we say that X has a nuclear Kothe quotients. It can also be shown that there always exists a Frechet space which has a nuclear Kothe subspace iff it has non-Banach subspace which admits continuous norm.

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INTRODUCTION

In 1936, M. Eidelheit showed that any non-Banach Frechet space has a quotient Isomorphic to ω.

In 1957, C. Bessaga and A.Pelczynski showed that a Frechet space fails to admit a continuous norm iff it has a subspace isomorphic to ω.

Again in 1959, Bessaga, Pelczynki and S.Rolewicz showed that a Frechet space which admits continuous norm has a nuclear kothe subspace iff it is not Banach. A Frechet space which admits continuous norm can have a quotient. If we consider only nuclear Frechet spaces, then we are looking for kothe quotients and the problem has a positive solution: every nuclear Frechet space not isomorphic to ω have a kothe quotients.

1. CONSTRUCTION OF A SPACE WITHOUT BASIS

Our characterization of Frechet spaces with nuclear kothe quotients will be in terms o the following condition which we lable(*) .If E is a Frechet space and (|| .||$_k$) a fundamental sequence of seminorms we denote by $E'_k(k \in N)$ the banach space determined by the unit ball of the dual norm $|| \, . \, ||'_k$.Moreover ,in view of the open mappings theorem ,the condition has the following equivalent formulation:

$$\exists\ l \ni\ \forall\ k \exists\ j\ \ni\ sup\{\ ||\ .\ ||'_k.: \mathrm{u} \in E'_l, ||\ .\ ||'_j \leq 1\} = \infty$$

In this form our condition is very close to being a dual to he following condition used by Bessaga, Pelczynski and Rolewicz in their determination of those Frechet spaces which have nuclear kothe subspaces.

$$\forall\ k\ \exists j \ni sup\{||x||_j: x \in Y, ||X||_k \leq 1\} = \infty$$

for every subspaces Y of E with Finite co dimension. The role of condition (*) in our characterization is contained in the following result.

$Proposition(1)$.

A Frechet space E satisfies condition (*) iff it has a quotient which admits continuous norm and satisfies condition (*).

$Proof$

Suppose That E satisfies condition (*).We may assume for this condition that $l = 1$ and $j = k\ + 1$, Let $M = (E'_1)^{00}$ which by the bipolar theorem ,is the closure of E'_1 in the weak topology from E .we will show that $\frac{E}{M^0}$ is the desired quotient.

First we check that $\frac{E}{M^0}$ admits continuous norm. Let $x \in E$ and suppose that the seminorm in $\frac{E}{M^0}$ induced by $||.||_1$ annihilates $x + M^0$.This means that

$$\exists (y_n) \subset M^0 \ni \lim_{n\to\infty} ||x + y_n||_1 = 0 .$$

Let $\in M$. Then $Ev \in E_1' \ni |u(x) - v(x)| \leq 1$

Hence we have

$$|u(x)| \leq |v(x)| + |u(x) - v(x)| \leq |v(x + y_n)| + 1$$

Since $v \in E_1'$ it follows that lim $v(x + y_n) = 0$ so $|u(x)| \leq 1$.

This shows that $x \in M^0$,so the seminorm induced by $||.||_1$ is a norm.

Now we verify condition (*) for $\frac{E}{M^0}$. Now we fix k and let V_k be the unit ball of $||.||_k$ in E . Snce E satisfies condition (*) we have a sequence $(u_n) \subset E_1'$ with $||u_n||_k'$ and $||u_n||_{k+1}' \leq \frac{1}{n}$ .This implies that $u_n \in \frac{1}{n} V_{K+1}^0 \sim V_k^0$ so a fortiori $u_n \notin (V_k + M^0)_0$.

Moreover , if $x \in V_{k+1}$,$y \in M^0$ then since $u_n \in M$

$$|nu_n(x + y) = |nu_n(x)| \leq 1$$

Hence $u_n \in \frac{1}{n}(V_{k+1} + M^c)_0$. Thus we have shown that

$u_n \in \frac{1}{n}(V_{k+1} + M^0)_0 \sim (V_k + M^0)_0$. But ($\left(\frac{E}{M^0}\right)'$ = M and the unit ball of the dual norm of the norm in $\frac{E}{M^0}$ induced by $||.||_k$ is $(V_k + M^0)^0$. This shows that $\frac{E}{M^0}$ satisfies condition (*).

Conversely let $\frac{E}{H}$ be a qutient of E. If ($V_k)$ is a fundamental sequence of nhds.of O for E then by general duality, $\left(\frac{E}{H}\right)'$ can be represented as a vector subspace of E' and a fundamental sequence of equicontinuous sets for($\frac{E}{M^0}$)'

Is given by $(V_k^0 \cap H^0)_k$. Hence if $\frac{E}{H}$ satisfies condition (*) (say with l and $j = k + 1)$ then $\exists (u_n) \subset H^0$ and a sequence of constants (C_n) with $u_n \in (C_n V_1^0 \cap H^0) \cap (\frac{1}{2} V_{k+1}^0 \cap H^0) \sim (V_k^0 \cap H^0)$.

It follows that $u_n \in E_1'$, $||u_n||_{k+1}' \leq \frac{1}{n}$ but since $u_n \in H^0$ then $u_n \notin V_k^o$ so $||u_n||_k' > 1$. Hence E satisfies condition (*).

Theorem (1)

A seperable Frechet space E has a nuclear Kothe quotient iff it satisfies condition (*).

Proof

Suppose that E satisfies condition (*) .By proposition (1) we may suppose that E admits continuous norm. Let ($d_n)$ be a dense sequence in E and E_0 the vector subspace it generates .we will establish the existence of a nuclear kothe quotient by constructing a biorthogonal sequence ($x_n, f_n)$.

Let $(||.||_k)$ be a fundamental sequence of norms for E .The space E_k' defined in the beginning of this section is the dual of the normed space ($E_0, ||.||_k$)

We may assume that condition (*) is satisfied with $l = 1$ and $j = k + 1$ .Then we have for each k$\in$N.

$sup\{||g||_k' / ||g||_{k+1}' : g \in E_1'\} = \infty$

Our construction is by induction. We can choose $f_1 \in E_1'$ with $f_1 \neq 0$.

Since E_0 is dense in E $\exists x_1 \in E_0 \ni f_1(x_1) = 1$.

Before passing to the induction .We must prove that the above statement of condition (*) remains true if the requirement $g \in E_1'$ is replaced by the weaker requirement that $g \in G$ where $\{G = g \in f_1 \in E_1' : g(x) = 0 , \forall x \in L\}$ for some finite $L \subset E$.

We may assume that L is a linearly independent set say $L = \{Z_v\}_{v=1}^m$.Since $||.||_1$ is a norm , E_1' is dense in E' with respect to the weak topology from E so we can find $\{g_v\}_{v=1}^m \subset E_1' \ni (Z_v, g_v)$ is biorthogonal. Then the map $P: E_1' \to E_1'$ defined by $Pf = \sum_{v=1}^m f(Z_v) g_v$ is a projection whose kernel is G and P is continuous for every $||.||_k'$.Hence if $H = P(E_1')$ we have $C_k > 0 (k \in N)$ with

$||g||'_k + ||h||'_k \leq C_k ||g+h||'_k$, $g \in G, h \in H)$

And , since H is finite dimensional,

$||h||'_k \leq C_k ||h||'_{k+1}$ (h$\in H)$

Therfore we have,for each $k \in N$,

$$\infty = sup \left\{ \frac{||f||'k}{||f||'k+1} : f \in E'_1 \right\} = sup \left\{ \frac{||g+h||'k}{||g+h||'k+1} : g \in G, h \in H \right\}$$

$$\leq C_{k+1} sup \left\{ \frac{||g||'k + ||h||'k}{||g||'k+1 + ||h||'k+1} : g \in G, h \in H \right\}$$

$$\leq C_{k+1} sup \left\{ \frac{||g||'k}{||g||'k+1} + \frac{||h||'k}{||h||'k+1} : g \in G, h \in H \right\}$$

$$\leq C_{k+1} sup \left\{ \frac{||g||'k}{||g||'k+1} : g \in G \right\} + C_k C_{k+1}$$

$Hence$ we may conclude that $sup \left\{ \frac{||g||'k}{||g||'k+1} : g \in G \right\} = \infty$

Turning to our induction step we assume that (x_i, f_i) (i=1,.........,$(n-1)$) have been selected. We apply the above condition with

$$G = \{g \in E'_1 : g(x_i) = g(d_i) = 0, i = 1,2, \dots \dots \dots n-1\}$$

Then we select $g_n, \dots \dots g_1$ inductively so as to annihilate $x_1, x_2, \dots \dots \dots . x_{n-1}$,

$d_1, d_2, \dots \dots \dots . d_{n-1}$, and satisfy, for $k = 1,2, \dots \dots .. n-1$.

$$||g_k||'_{k+1} \leq \frac{1}{2^k}$$,

$$||g_k||'_{k+1} > (n^2+1)(1 + ||g_{k+1}||'_k + \cdots \dots \dots . . ||g_k||'_k$$

$Next$ we set $f_n = g_1 + g_2, \dots \dots \dots . + g_n \in E'_1$

$So\ f_n$ annihilates x_i, $d_i(i = 1, \dots \dots \dots . n-1,$

$$\frac{||f_n||'_{k+1}}{||f_n||'_k} \leq \frac{||g_1||'_{k+1} + \cdots + ||g_k||'_{k+1} + ||g_{k+1}||'_{k+1}| + \cdots . + |g_n||'_{k+1}}{-(||g_1||'_k + \cdots ||g_{k-1}||'_k) + ||g_n||'_k - (||g_{k+1}||'_k + \cdots . + + ||g_n||'_k}$$

$$< \frac{\frac{1}{2} + \cdots . + \frac{1}{2^k} + ||g_{k+1}||'_k + \cdots + ||g_n||'_{k+1}}{-1 + (1+n^2)(1 + \cdots \dots + ||g_{k+1}||'_k + \cdots . + ||g_n||'_k)}$$

$< \frac{1 + ||g_{k+1}||'_{k+1} + \cdots . + ||g_n||'_{k+1}}{n^2(1 + ||g_{k+1}||'_k + \cdots + ||g_n||'_k} \leq \frac{1}{n^2}$ as desired.

$Finallly$ since E_0 is dense in E and $f_n \neq 0$ we can find

$x_n \in E_0 \ni f_n(x_n) = 1$ and $f_i(x_n) = 0\ (i = 1, \dots \dots , n-1), thus\ the\ i$nduction is completed.

$Proposition(2)$

A Frechet space E is a quojection iff every quotient of E which admits continuous norm is a Banach space.

Proof

Let E be a quojection and f be a quotient of E. If E is the projective limit of the surjections $A_k : E_{k+1} \to E_k$ then it is easy to see from the definition of projective limit that the canonical projections $P_k : E \to E_k$(given by $P_k((x_j)_j) = x_k)$ are also surjections .Let $T : E \to F$ be the quotient map,|.| a continuous norm on F and $||.||_k$the norm on E_k.

By the ontinuity of $T \exists k$ and $C > 0 \ni |Tx| \leq C||P_k x||_k$,x$\in$ E.Hence from an algebraic point of view $\exists$ is a unique map $S : E_k \to F \ni SP_k = T$. Since T is surjection S is also .Moreover ,it follows from the losed graph theorem and the fact that |.| is a norm that S is continuous.Hence ,F is a quotient of a Banach space so it is a Banach space.

Coversely let $(||.||_k)$ be aa fundamental sequence of seminorms for E ,$N_k$the kernel of $||.||_k$ and $\frac{E}{N_k}$ the quotient Fechet space. Since $(||.||_k)$ is increasing we have the canonical maps $\frac{E}{N_{k+1}} \to \frac{E}{N_k}$ which are surjections and it is easy to see that E is isomorphic to the projective limit of this sequence of maps.On the other hand ,the seminorm induced on $\frac{E}{N_k}$ by $(||.||_k)$ is clearly a norm so by assumption , $\frac{E}{N_k}$ is aBanach space.Thus ,E is aquojection.

Corollary(1)

If E is a quojection then E does not have a nuclear Kothe quotient.

Corollary (2)

If E is isomorphic to a countable product of Banach space then E does not have a nuclear kothe quotient.

It would be nice to know that the quojections are precisely those Frechet spaces (amongst separable spaces) which fail to have nuclear kothe quotients .

It would be nice to know that the quojections are precisely those Frechet spaces(amongst separable spaces) which fail to have nuclear kothe quotients .Unfortunately ,we are unable to prove the converse of corollary 1 so this remains open. We can obtain this result ,however ,if we restrict our considerations to reflexive spaces.

In order to investigate this situation we consider , for an arbitrary Frechet spaces E, the vector space E'^b of all linear functional on E' which are bounded on bounded sets .Obviously $< E' , E'^b >$ is a dual system and we indicate thepolar of a set A by A^b. We call consider the topology τ on E'^b of uniform convergence on bounded sets,that is,a fundamental system of neighbourhood of o is given by the sets B^b, B is a bounded subset of E'.

Proposition (3)

If E is a separable Frechet space which does not have a nuclear Kothe quotient then E'^b is quojetion.

Proof

We can apply theorem(2) to conclude that E does not satisfy condition(*) and so we can find a fundamental sequence of seminorms $(||.||_k)$ for $E \ni$ for every k, the $||.||'_{k+1}$ closure of E'_k is closed in each Banach space E'_j, $j > k+1$.This implies that if $F_k$is the $||.||'_{k+1}$closure of E'_k in E'_{k+1} and each F_k is equipped with the norm $||.||'_{k+1}$ then F_k is closed subspace of F_{k+1}.

But the unit balls of the $F_k(k \in N)$ form a fundamental sequence of bounded sets for E' and it is easy to check that E'^b is isomorphic to the projective limit of the sequence of maps $F'_{k+1} \to F'_k$, adjoint to the inclusions $F_k \to F_{k+1}(k \in N)$.Since F_k is a closed subspace of F_{k+1}it follows that the maps $F'_{k+1} \to F'_k$are surjections so E'^b is quojetion.

$corollary\ (3)$

If E is a separable reflexive Frechet space,then E has a nuclear Kothe quotient iff E is not a quojection.

Corollary(4)

Every Frechet Montel space not isomorphic to ω has a nuclear kothe quotient.

Proof

If E is a Frechet Montel space then, as is well known ,E is separable and reflexive. We will show that E is not a quojection. Suppose that E is the projective limit of the surjections $A_k: E_{k+1} \to E_k$. We may assume that A_k maps the unit ball of E_{k+1}onto the unit ball of E_k.Also ,since E is not isomorphic to ω we may assume that E_1 is infinite dimensional. Then, viewing E as the projective limit, it is easy to see that $\{(x_k) \in E: x_k \text{ is the unit ball of} E_k \forall k\}$ is a closed,boundedsubset of E.But the projection of this set in E_1 is the unit ball so it is not compact.Hence ,the set is not compact in E which is a contradiction.

Frechet spaces which admits continuous norm.Again the situation with quotients seems more difficult than in the case of subspaces.We do not know whether a separable non-banach Frechet spac which admits a continuous norm (or is even countable normed) necessarily has nuclear kothe quotient. Actually, this is the same as the above question regarding the converse of corollary (1). In fact, if E is not a quojection, then by proposition (2), E has a non-Banach quotient which admits a continuous norm, so if this question had a positive answer, E would have a nuclear kothe quotient and thus the converse of corollary (1) would hold. Conversely, if E is a non-Banach Frechet space which admits continuous norm then clearly E is not a quojection so if the converse of corollary (1) held ,E would have nuclear kothe quotient.

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