Journal of Advances and Scholarly Researches in Allied Education
Vol. 16, Issue No. 6, May-2019, ISSN 2230-7540

A Study on Neutral Differential Functional Equation in Banach Algebras

Annu Yadav[1]* Dr. Ashwani Kumar[2]

[1] Research Scholar, Sunrise University, Alwar, Rajasthan

[2] Associate Professor, Sunrise University, Alwar, Rajasthan

Abstract – In this paper, the existence theorem for the first order functional differential equations in Banach algebras is seen under the mixed generalized conditions of Lipschitz and Caratheodory. The presence of Extremal solutions is also proven under such monotonicity conditions.

Key Words – Differential Equations, Banach Algebras, Extremal Solutions

---------------------------------X---------------------------------

INTRODUCTION

In the first order of practical differential equations, the presence theorem is seen in the mixed generalized condition of Lipschitz and caratheodorum. There are also some monotonic parameters for severe solutions.

Statement of Problem

Let the R actual line be labelled and let I0 = [-r, 0] and *I* = [0, a] be two cycles closed and confined . Let $J = I_0 \cup I$, then. It's an interval closed and limited. Let C mean the Banach field of all continuous functions that are valued Ø on I_0 with the best level ". $\|_C$ defined by

$$\| \emptyset \|_C = \sup_{t \in I_0} |\emptyset(t)|.$$

Clearly C is a Banach algebra with this norm. Consider the first order functional differential equation (in short FDE)

$$\left. \begin{aligned} &\frac{d}{dt}\left[\frac{x(t)}{f(t, x(t), x_t)}\right] = g(t, x(t), x_t)\ a.e.\ t \in I \\ &x(t) = \emptyset(t), t \in I_0 \end{aligned} \right\}$$

Where
$f : I \times \mathbb{R} \times C \to \mathbb{R} - \{0\},\ g : I \times \mathbb{R} \times C \to \mathbb{R}$
and for each t C *I* , $x_t : I_0 \to C$ is a continuous function defined by

$$x_t(\theta) := x(t + \theta)\ for\ all\ \theta \in I_0 .$$

By a solution of FDE we mean a function $x \in C(J, \mathbb{R}) \cap AC(I, \mathbb{R}) \cap C(I_0, \mathbb{R})$ that satisfies the equations, where *AC(I,)* is the space of all absolutely continuous real- valued functions on *J*.

The most productive field of study has been practical differential equations for a long time. The connection therein can be found in Hale (1977), Henderson (1995). But in Banach algebra the examination of functional differential equations is very rare. Rather recently, this analysis was begun utilizing fixed point theorems. See the sources therein for Dhage and Regan (2000) and Dhage (1999). The FDE is recent and would undoubtedly make an enormous contribution to the field of functional differential equations by researching it. The fixed point theorem is given in the section below.

Auxiliary Results

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping *A : X ‹ X* is called Ð-Lipschitz if there exists a continuous nondecreasing function ψ : + ‹ + Satisfying

$$\| Ax - Ay \| \le \psi \ \| x - y \|$$

for all $x, y \in X$ with ψ (0) = 0. In the special case when ψ (r) = αr (α > 0), A is called a Lipchitz with a Lipschitz constant α. In particular, if α < 1, A is called a contraction with a contraction constant α. Further, if ψ ( ) € for all Σ 0, then *A* is called a nonlinear contraction on *X*. Sometimes we call the function ψ a Đ- function for convenience.

If compact, an operator T : X → X is referred to as compact if $\overline{T(X)}$ It is a compact X subset. Similarly, if T maps a bounded subset of X into a sufficiently compact subset of X, T : X → X is considered totally bounded. Finally, if it is a continuous and absolutely bounded operator on X, T : X → X is considered an entirely continuous operator. It is apparent that any compact operator is absolutely constrained, but the reverse might not be valid. The Schaefer style nonlinear alternative recently shown by Dhage (2005-c) is expressed in the following theorem.

Theorem (Dhage ,2005-c) Let X be a Banach algebra and let

A, B : X → X be two operators satisfying

a) A is a Đ -Lipschitz with a Đ -function ψ,

b) B is compact and continuous, and

c) M ψ (r) < r whenever r > 0, wherever

M = " B(X) " $x \,\|\, x \in X$}, then either

(i) The equation λ $\left(\frac{x}{\lambda}\right) Bx = x$ has a solution for λ = 1, or

(ii) The set C = {u C X | λA $\left(\frac{u}{\lambda}\right)$ Bu = u, 0 € λ€ 1} is unbounded.

It is known that Theorem useful for proving the existence theorems for the integral equations of mixed type. See Dhage (1994) and the references therein. The method is commonly known as priori bound method for the nonlinear equations. See, for example, Dugundji et.al. (1982), Zeidler (1985-b) and the references therein.

In its relevant form, an interesting corollary to Theorem is the corollary Let X be a Banach algebra and let A, B:X → X be two operators Satisfying.

(a) A is Lipschitz with a Lipschitz constant α,

(b) B is compact and continuous, and

(c) α M < 1 , where M =" B(X)" := sup{Bx": xCX}, So either So the balance ßA $\left(\frac{x}{\beta}\right) Bx = x$ has an explanation for ß = 1, or the set $\varepsilon = \{u \in X \,|\, \lambda A\left(\frac{u}{\lambda}\right) Bu = u, 0 < \lambda < 1\}$ is unbounded. Existence Theory

May M (J, %) and B(I, %) represent the observable and minimal real-time function spaces in J, respectively. We shall search for an FDE solution for all totally continuous re-appraised functionalities on J in space C(J, %). Set "•" standard in C (J,%) by

$$\| x \| = \sup_{t \in J} |x(t)|.$$

Clearly, C(J, %) with this norm becomes a Banach algebra. Please notice that C(J,%)ŠAC(J,%)). The following concept is needed in the sequel.

Definition: A cartography ß: I × % × C‹ % The state of Caratheodory or merely Caratheodory can be said to fulfill, whether

(i) for any x C C, t→ß (t, x , y) can be calculated and

(ii) x → ß for t CI — is virtually everywhere continuous (t, x , y).

(iii) Once again the feature ß (t, x , y) is referred to as L1-Carathion if each amount in real terms r Σ 0 A function exists hr C L^1(I,%) such that

$$|ß(t, x, y)| = h_r(t), \text{ a.e. } t \text{ C } I.$$

for all x C % and y C with |x| € r and " y "C € r.

Finally a Caratheodory function ß(t, x, y) is called L^1_x -

Caratheodory if

(i) there exists a function $\in L^1(I,\mathbb{R})$ such that $|ß(t,x,y)| = < h(t), a.e. t \in I$ for all $x \in \mathbb{R}$ and $y \in C$.

For convenience, the function h is referred to as a bound function of ß.

We will need the following hypotheses in the sequel.

(H1) The functionality $f : I \times \% \times C < \%$ is continuous and a function occurs k C B(I, %) such that $k(t) > 0$, a.e. t C I and $|f(t, x1, y1) - f(t, x2, y2)| \le k(t)\max\{|x1-x2|, "y1-y2$ for all x1, x2 C %.

(H2) $f(0, \emptyset(0), \emptyset) = 1.$

(H3) Function (g) L^1_x-Carathodory Bound with h feature. (H4) There exists a continuous and non-decreasing function $\Omega: [0,\infty) \to (0,\infty)$ and a function $\gamma \in L^1(I,\mathbb{R})$ such that γ(t) Σ 0, a.e. t C J and $|g(t,x,y)| \le \gamma(t)\,\Omega\, max\{|x|, \| y \|_C\}, a.e. t \in I,$ for all x C C .

Theorem Assume that the hypotheses(H1)-(H4) hold. Suppose that

$$\int_{C_1}^{\infty} \frac{ds}{\Omega(s)} > C_2 \| \gamma \|_{L^1},$$

Where

$$C_1 = \frac{F \| \emptyset \|_C}{1 - \| k \| (\| \emptyset \|_C + \| h \|_{L^1})}, C_2 = \frac{1}{1 - \| k \| (\| \emptyset \|_C + \| h \|_{L^1})},$$

$$\| k \| (\| \emptyset \|_C + \| h \|_{L^1}) < 1. F = \max_{t \in J} |f(t,0,0)|, \text{ and } \| k \| = \max_{t \in J} |k(t)|,$$

then the FDE has a solution on J.

Proof. The FDE is now analogous to the functional integral equation (FIE in short),

$$x(t) = f[t, x(t), x_t)] \left(\emptyset(0) + \int_0^t g(s, x(s), x_s)\, ds \right), if\ t \in I.$$

And

$$x(t) = \emptyset(t), if\ t \in I_0.$$

Defines the two mappings of C (J, %) by A and B

$$Ax(t) = \begin{cases} f(t, x(t), x_t), & if\ t \in I. \\ 1. & if\ t \in I_0. \end{cases}$$

And

$$Bx(t) = \begin{cases} \emptyset(0) + \int_0^t g(s, x(s), x_s) ds, & if\ t \in I. \\ \emptyset(t), & if\ t \in I_0, \end{cases}$$

Obviously A and B define the operators A,B:C(J,%)‹C(J,%).

Then the FDE is equivalent to the operator equation

$$x(t) = Ax(t)\, Bx(t), t \in J.$$

We would illustrate that operators A and B follow all Corollary hypotheses.

We first show that A is a Lipschitz on C(J, %).

Let x, y CC(J, %). Then by(H1),

$$|Ax(t) - Ay(t)| \le |f(t, x(t), x_t) - f(t, y(t), x_t)|$$

$$\le k(t) \max \{|x(t) - y(t)|, \text{"}x_t - y_t\text{"}_c\}$$

$$\le \| k \| \| x - y \|$$

for everyone t C J. We get the dominance over

$$\| Ax - Ay \| \le \| k \| \| x - y \|,$$

for all x,y C C(J, %). Therefore A is a Lipschitz with a constant "k" on C(J,%) Next we demonstrate that on C(J, %, B is entirely continuous). Usage of the norm as in Granas et. al.(1991) B is a constant C(J, %) generator. Enable S to be a small collection in C (J,%). It will be shown that B(C(J,%) is an universally small and equicontinuous range in C(J,%). Since g(t, x(t) is an xt), we have

$$|Bx(t)| \le \| \emptyset \|_C + \int_0^t |g(s, x(s), x_s)| ds$$

$$\le \| \emptyset \|_C + \int_0^t h(s) ds$$

$$\le \| \emptyset \|_C + \| h \|_{L^1}$$

Taking the supremum over t, we obtain "Bx" ≤ M for all x C S, where $\|_{L^1}$. This indicates that B(C (J,%) is an universally restricted range in C(J,%). We now demonstrate that B(C (J,%) is fitted. Let t, ć I. So for everyone x C C (J , %) we have,

$$|Bx(t) - Bx(\tau)| \le \left| \int_0^t g(s, x(s), x_s) ds - \int_0^\tau g(s, x(s), x_s) ds \right|$$

$$\le \left| \int_\tau^t |g(s, x(s), x_s)| ds \right|$$

$$\le \left| \int_\tau^t h(s) ds \right|$$

$$\le |p(t) - p(\tau)|$$

Where

$$p(t) = \int_0^t h(s)\, ds.$$

Therefore

$$|Bx(t) - Bx(\tau)| \to 0 \text{ as } t \to \tau.$$

Again let τ C I0, t C I . We'll then have

$$|Bx(t) - Bx(\tau)| \le |\emptyset(\tau) - \emptyset(0)| + \left| \int_0^t g(s, x(s), x_s) ds \right|$$

$$\le |\emptyset(\tau) - \emptyset(0)| + |p(t) - p(\tau)|$$

where the function p is defined above. Similarly if τ, t C I0, then we get

$$|Bx(t) - Bx(\tau)| \le |\emptyset(t) - \emptyset(\tau)|.$$

Therefore in all above three cases

$$|Bx(t) - Bx(\tau)| \to 0 \text{ as } \tau \to t,\ \forall t,\tau \in J.$$

Hence B(C (J,%)) is an equicontinuous set and consequently B(C (J, %)) is relatively compact by Arzela-Ascoli theorem. As a result B is a compact and continuous operator on C(J, %). Both the requirements of the theorem are then met and a subsequent implementation of it results in either the statement I or the conclusion (ii) being retained. We illustrate that it is not possible to assume (ii). Let x C X be any FDE solution. Then we have, for any λ C (0, 1),

$$x(t) = \lambda A\left(\frac{x}{\lambda}\right)(t)Bx(t)$$

$$= \begin{cases} \lambda\left[f\left(t,\frac{x(t)}{\lambda},\frac{x_t}{\lambda}\right)\right]\left(\phi(0)+\int_0^t g(s,x(s),x_s)\,ds\right), t\in I \\ \lambda\phi(t), t\in I_0\end{cases}$$

for t C J. So if t C I0, So we've got

$$|x(t)| \le \lambda\|\emptyset\|_C \le \|\emptyset\|_C.$$

Again if t C I, So we've got

$$|x(t)| \le \lambda\left|f\left(s,\frac{x(t)}{\lambda},\frac{x_t}{\lambda}\right)\right|\left(\|\emptyset\|_C+\left|\int_0^t g(s,x(s),x_s)\,ds\right|\right)$$

$$\le \lambda\left(\left|f\left(s,\frac{x(t)}{\lambda},\frac{x_t}{\lambda}\right)-f(t,0,0)\right|+|f(t,0,0)|\right)$$

$$\times\left(\|\emptyset\|_C+\int_0^t|g(s,x(s),x_s)|\,ds\right)$$

$$\le [k(t)\max\{|x(t)|,\|x_t\|_C\}+F]\left(\|\emptyset\|_C+\int_0^t|g(s,x(s),x_s)|\,ds\right)$$

$$\le k(t)\max\{|x(t)|,\|x_t\|_C\}\left(\|\emptyset\|_C+\int_0^t|g(s,x(s),x_s)|\,ds\right)$$

$$+F\left(\|\emptyset\|_C+\int_0^t|g(s,x(s),x_s)|\,ds\right)$$

$$\le \|k\|\max\{|x(t)|,\|x_t\|_C\}(\|\phi\|_C+\|h\|_{L^1})+F\|\phi\|_C$$

$$+F\int_0^t\gamma(s)\Omega(\max\{|x(t)|,\|x_s\|_C\})\,ds.$$

Put $u(t) = |x(s)|$, for t C J. So we had max $\{|x(t)|, "xt"C\} \le u(t)$ for all t C J, There's a point, then t* C [-r, t] such that $u(t) = |x(t^*)|$. This is the result of

$$u(t) = |x(t^*)|$$

$$\le \|k\|\,|x(t^*)|(\|\phi\|_C+\|h\|_{L^1})$$

$$= C_1 + C_2\int_0^t\gamma(s)\Omega(u(s))\,ds$$
$$+F\left(\|\phi\|_C\int_0^t\gamma(s)\Omega(\max\{|x(t)|,\|x\|_C\})\,ds\right)$$

$$\le \|k\|u(t)(\|\phi\|_C+\|h\|_{L^1})+F\left(\|\phi\|_C\int_0^t\gamma(s)\Omega(u(s))\,ds\right)$$

Where

$$C_1=\frac{F\|\phi\|_C}{1-\|k\|[\|\phi\|_C+\|h\|_{L^1}]} \text{ and } C_2=\frac{1}{1-\|k\|[\|\phi\|_C+\|h\|_{L^1}]}$$

Let

$$\omega(t)=C_1+C_2\int_0^t\gamma(s)\Omega(u(s))\,ds.$$

Then u(t) ≤ m(t) and a direct differentiation of m(t) yields

$$\left.\begin{aligned}\omega'(t)&\le C_2\gamma(t)\Omega(\omega(t))\\ \omega(0)&=C_1\end{aligned}\right\}$$

that is

$$\int_0^t\frac{\omega'(s)}{\Omega(\omega(s))}\,ds\le C_2\int_0^t\Upsilon(s)\,ds\le C_2\|\gamma\|_{L^1}$$

A shift in the above integral variables means that

$$\int_{C_1}^{w(t)}\frac{ds}{\Omega(s)}\le C_2\|\gamma\|_{L^1}<\int_{C_1}^{\infty}\frac{ds}{\Omega(s)}.$$

Now an application of mean value theorem yields that there is a constant M > 0 such that ω(t) ≤ M for all t C J. This further implies that

$$|x(t)|\le u(t)\le\omega(t)\le M.$$

for all t C J. Thus the conclusion (ii) of Corollary does not hold. Therefore the operator equation Ax Bx = x and consequently the FDE has a solution on J. This completes the proof.

REFERENCES

[1] D. W. Boyd and S. W. Wong (1969). On nonlinear contractions. Proc. Amer. Math. Soc. 20, pp. 456-464.

[2] F. E. Browder (1973). Nonlinear operators and nonlinear equations of evolution in Banach spaces. Nonlinear functional analysis, Proc. Sym. Pure Math. Vol. 18, Part II, Amer. Math. Soc.

[3] T. A. Burton and C. Kirk (1998). A fixed point theorem of Krasnoselskii-Schaefer type, Math. Nachr. 189. pp. 23-31.

[4] B. C. Dhage (1994). On a - condensing mappings in Banach algebras. The Math. Student 64, pp. l46-152.

[5] B. C. Dhage, On some nonlinear alternatives of Lera y - Schauder type with applications to functional integral equations. I. Arch. Math. (Brno) in press.

[6] B. C. Dhage (1988). On some variants of Schauder's fixed Point principle and applications, to nonlinear integral equations. 1. Math. Phy. Soc. 25, pp. 603-611.

[7] B. C. Dhage, R. N. Ingle and S. K. Ntouyas (2004). Existence of extremal random solutions of nonlinear random integral equations. Communications on Applied Nonlinear Analysis, volume 11, No. 4, pp. 43-56.

[8] B. C. Dhage and R. N. Ingle (2007). Nonlinear functional random differential equations in Banach algebras. Tamkang Journal of mathematics, vol. 38, No. 1, pp. 57-73.

[9] A. Friedman (1975). Stochastic Differential Equations and Applications. Vol. I and II. Academic Press, New York.

[10] A. Granas, R. B. Guenther, and 1. W. Lee (1991). Some general existence principles in the Caratheodory theory of nonlinear differential equations. 1. Math. Pure. et. Appl. 70, pp. 153-196.

Corresponding Author

Annu Yadav*

Research Scholar, Sunrise University, Alwar, Rajasthan