

# A Study on the Reduced Differentials Transforms Approach for Nonlinear Partial Integro-Differential Equations

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**Abstract** – In this study, we have studied the reduced differential transforms approach for nonlinear partial integrodifferential equations in which we discussed Strict and Non-Strict Inequalities, Existence Results, Maximal and Minimal Solutions, and then Comparison Theorems.

**Keywords** – Differential, Inequalities, Comparison Theorems

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## INTRODUCTION

Given a bounded interval  $J = [t_0, t_0 + a)$  in  $\mathbb{R}$  for some fixed  $t_0, a \in \mathbb{R}$  with  $a > 0$ , with  $a > 0$ , Consider the hybrid differential hybrid value issues (in short HIDE),

$$\left. \begin{aligned} \frac{d}{dt}[x(t) - f(t, x(t))] &= \int_{t_0}^t g(s, x(s)) ds, \quad t \in J \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned} \right\}$$

Where,  $f, g : J \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous.

By a solution of the HIDE (5.1.1) we mean a function  $x \in C(J, \mathbb{R})$  such that

- (i) The function  $t \mapsto x - f(t, x)$  is continuous for each  $t \in \mathbb{R}$ , and
- (ii)  $x$  satisfies the equations in.

The significance of research into HIEs comes in the fact that many classes of differential and integrative equations are included in these instances as special cases. In the Krasnosel ski work, the subject of hybrid differential equations is included and is thoroughly examined on hybrid differential equations with various disturbances in the many publications. For the references within, see Burton, Dbage. This class of hybrid integrate differential equations involves the disturbances in various methods of the original integral differential equations. Dhage offers a precise categorization of various kinds of disturbances of integral differential equations, which may be handled using hybrid Dhage, Dhage, and Lai—Shmikantham fixed point theory. In this section, we begin with the basic theory of the second

nonlinear hybrid, whole-difference mixed disruptive equations, and show some basic findings, such as integrated inequality, theorem of existence, maximum and minimum solutions. We note that the results of this chapter add to the theory of nonlinear normal integral and differential equations in principle and substantially.

## STRICT AND NON-STRICT INEQUALITIES

The following hypothesis is often necessary.

(Ao) The function  $x \mapsto x - f(t, x)$  is increasing in  $\mathbb{R}$  for all  $t \in J$ ,

We are first of all demonstrating the fundamental findings of hybrid, integral inequality.

**Theorem 1.1** Assume that the hypothesis (Ao) holds. Suppose that they exist

$y, z \in C(J, \mathbb{R})$  such that

$$\frac{d}{dt}[y(t) - f(t, y(t))] \leq \int_{t_0}^t g(s, y(s)) ds, \quad t \in J,$$

And

$$\frac{d}{dt}[z(t) - f(t, z(t))] \geq \int_{t_0}^t g(s, z(s)) ds, \quad t \in J.$$

If one of the inequalities is strict and

$$y(t_0) < z(t_0),$$

then

$$y(t) < z(t)$$

or all  $t \in J$ .

**PROOF.** Suppose the unfairness is wrong. Then the set P is set to

$$P = \{t \in J \mid y(t) \geq z(t)\}$$

is non-empty.  $t_1 = \inf P$ . We can assume that without losing the generality

$$y(t_1) = z(t_1) \text{ and } y(t) < z(t)$$

for all  $t < t_1$ .

Assume that

$$\frac{d}{dt}[z(t) - f(t, z(t))] > \int_0^t g(s, z(s)) ds$$

for  $t \in J$ .

Denote

$$Y(t) = y(t) - f(t, y(t)) \text{ and } Z(t) = z(t) - f(t, z(t))$$

for  $t \in J$ .

As hypothesis (Ao) holds, it follows from that

$$Y(t_1) = Z(t_1) \text{ and } Y(t) < Z(t)$$

for all to  $t < t_1$ . The above relation further yields

$$\frac{Y(t_1 + h) - Y(t_1)}{h} > \frac{Z(t_1 + h) - Z(t_1)}{h}$$

Little  $h < 0$ . We get the limit as  $h \rightarrow 0$ .

$$Y'(t_1) \geq Z'(t_1).$$

That's why we receive

$$\int_{t_0}^{t_1} g(s, y(s)) ds \geq Y'(t_1) \geq Z'(t_1) > \int_{t_0}^{t_1} g(s, z(s)) ds.$$

The evidence is full. This is a contradiction.

The second finding is that the HIDE on J does not have a unilateral Lipschitz status.

### EXISTENCE RESULT

In this section, we prove an existing result for the HIDE on a closed and bounded interval  $J = [t_0, t_0+a]$  under mixed Lipschitz and compactness conditions on the nonlinearities involved in it. We place the HIDE in the space  $C(J, M)$  of continuous real-valued functions defined on J and use a hybrid fixed point of Dhage. Define a supremum norm  $\| \cdot \|$  in  $C(J, R)$  defined by

$$\|x\| = \sup_{t \in J} |x(t)|.$$

$C(J, R)$  is a space in Banach with the supremum standard above. The solution to the HDE is shown in the Banachspace by Dhage through the hybrid fixed point theorem. We provide certain preliminaries and definitions before specifying the fixed point theorem, which will then be utilized.

**Definition 1:** A mapping  $\psi: R_+ \rightarrow R_+$  is called a dominating function or, in short, D-function, if it is upper semi-continuous and non-decreasing function satisfying  $\psi(r) = kr$  and  $\psi(r) = \frac{r}{1+r}$  etc. These D-functions have been widely used in the theory of nonlinear differential and integral equations for proving the existing results via fixed point methods.

$$\|Q\phi - Q\xi\| \leq \psi(\|\phi - \xi\|)$$

The details of different types of contractions appear in monographs of Dhage and Granas and Dugundji. There do exist D-functions and the commonly used D-functions are  $\psi(r) = kr$  and  $\psi(r) = \frac{r}{1+r}$  etc. These D-functions have been widely used in the theory of nonlinear differential and integral equations for proving the existing results via fixed point methods.

**Definition 2:** An operator Q on a Banach space E into itself is called compact if  $Q(E)$  is a relatively compact subset of E. Q is called bounded if, for any bounded subset S of E,  $Q(S)$  is a relatively compact subset of E. If Q is continuous and bounded, then it is called completely continuous on E.

**Theorem 1.2:** Suppose that S is a closed, convex, and bounded subset of the Banach space E and let  $A: E \rightarrow E$  and  $B: S \rightarrow E$  be two operators such that

- (a) A is nonlinear contraction,
- (b) B is compact and continuous, and
- (c)  $x = Ax + By$  for ally  $y \in S \Rightarrow x \in S$

Then the operator equation  $Ax + Bx = x$  has a solution in  $S$ .

**We consider the following hypotheses in what follows.**

(A1) There exists a constant  $L > 0$  such that

$$|f(t, x) - f(t, y)| \leq \frac{L|x - y|}{M + |x - y|}$$

for all  $t \in J$  and  $x, y \in \mathbb{R}$ . Moreover,  $L < M$ .

(A2) There exists a continuous function  $h: J \rightarrow \mathbb{R}$  such that

$$|g(t, x)| \leq h(t), \quad t \in J$$

For all  $x \in \mathbb{R}$ .

The following lemma is useful in the sequel.

**Lemma 1.1:** Assume that hypothesis (AO) holds. Then for any continuous function  $h: J \rightarrow \mathbb{R}$ , the function  $x \in C(J, \mathbb{R})$  is a solution of the HIDE

$$\left. \begin{aligned} \frac{d}{dt}[x(t) - f(t, x(t))] &= \int_{t_0}^t h(s) ds, \quad t \in J \\ x(0) &= x_0 \in \mathbb{R} \end{aligned} \right\}$$

if and only if  $x$  satisfies the hybrid integral equation (HIE)

$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \int_{t_0}^t (t-s)h(s) ds, \quad t \in J.$$

**PROOF.** Let  $h \in C(J, \mathbb{R})$ . Assume first that  $x$  is a solution of the HIDE. By definition,  $x(t) - f(t, x(t))$  is continuous on  $J$ , and so, differentiable there, where  $\frac{d}{dt}[x(t) - f(t, x(t))]$  is integrable on  $J$ . Applying integration from  $t_0$  to  $t$ , we obtain the HIE on  $J$ .

On the other hand, imagine that  $x$  meets HIE. Then we get the first equation in by straight differentiation. Again,  $t = t_0$  to replace in returns

$$x(t_0) - f(t_0, x(t_0)) = x_0 - f(t_0, x_0).$$

Since the mapping  $x \mapsto x - f(t, x)$  in  $K$  for every  $t \in J$ , mapping increases  $x \mapsto x - f(t_0, x)$  is injective in  $\mathbb{R}$ , whence  $x(t_0) = x_0$ . The lemma is therefore fully proven.

Now we can demonstrate the following HIDE existence theorem at  $J$ .

## MAXIMAL AND MINIMAL SOLUTIONS

In this section, we shall prove the existence of maximal and minimal solutions for the HIDE on  $J = [t_0, t_0 + a]$ . We need the following definition in what follows.

**Definition 2:** A solution  $r$  of the HIDE is said to be maximal if for any other solution  $x$  to the HIDE one has  $x(t) < r(t)$ , for all  $t \in J$ . Again, a solution  $p$  of the HIDE is said to be minimal if  $p(t) < x(t)$ , for all  $t \in J$ , where  $x$  is any solution of the HIDE existing on  $J$ .

The case for the maximum solution is discussed mainly because the minimum solution is comparable and with suitable amendments may be reached with similar reasoning. Consider the following initial value issue of hiding, given an arbitrarily small real number  $\epsilon > 0$ ,

$$\left. \begin{aligned} \frac{d}{dt}[x(t) - f(t, x(t))] &= \int_0^t g(s, x(s)) ds + \epsilon, \quad t \in J \\ x(t_0) &= x_0 + \epsilon \end{aligned} \right\}$$

where,  $f, g \in C(J \times \mathbb{R}, \mathbb{R})$ .

A HIDE existence theorem can be described as follows:

**Theorem 1.3** Assume that the (AO) by (A2) assumptions are in effect. The HIDE has a solution specified on  $J$  for any small integer  $\epsilon > 0$ .

**PROOF.** Let  $\epsilon_n$  be such a declining series of good real numbers  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ . Then, by Theorem, one obtains any solution of the HIDE

$$u(t) < r(t, \epsilon_n)$$

for all  $t \in J$  and  $n \in \mathbb{N} \cup \{0\}$ , where  $r(t, \epsilon_n)$  is a solution of the HIDE,

$$\left. \begin{aligned} \frac{d}{dt}[x(t) - f(t, x(t))] &= \int_0^t g(s, x(s)) ds + \epsilon_n, \quad t \in J \\ x(t_0) &= x_0 + \epsilon_n \end{aligned} \right\}$$

Defined on  $J$ .

Since, by Theorems,  $r(t, \epsilon_n)$  is a reduction of positive real numbers, the limit  $r(t) = \lim_{n \rightarrow \infty} r(t, \epsilon_n)$  exists.

We show that the convergence is uniform on  $J$ . To finish, it is enough to prove that the sequence  $\{r(t, \epsilon_n)\}$  is equip continuous in  $C(J, \mathbb{R})$ . Let  $t_1, t_2 \in J$  be arbitrary. Then,

$$\begin{aligned}
 & |r(t_1, \epsilon_n) - r(t_2, \epsilon_n)| \\
 & \leq |f(t_1, r(t_1, \epsilon_n)) - f(t_2, r(t_2, \epsilon_n))| \\
 & \quad + \left| \int_{t_0}^{t_1} (t_1 - s)g(s, r_{\epsilon_n}(s)) ds - \int_{t_0}^{t_2} (t_2 - s)g(s, r_{\epsilon_n}(s)) ds \right| \\
 & \quad + \left| \int_{t_0}^{t_1} (t_1 - s)\epsilon_n ds - \int_{t_0}^{t_2} (t_2 - s)\epsilon_n ds \right| \\
 & \leq |f(t_1, r(t_1, \epsilon_n)) - f(t_2, r(t_2, \epsilon_n))| \\
 & \quad + \left| \int_{t_0}^{t_1} (t_1 - s)g(s, r_{\epsilon_n}(s)) ds - \int_{t_0}^{t_2} (t_2 - s)g(s, r_{\epsilon_n}(s)) ds \right| \\
 & \quad + \left| \int_{t_0}^{t_2} (t_2 - s)g(s, r_{\epsilon_n}(s)) ds - \int_{t_0}^{t_2} (t_2 - s)g(s, r_{\epsilon_n}(s)) ds \right| \\
 & \quad + \left| \int_{t_0}^{t_1} (t_1 - s)\epsilon_n ds - \int_{t_0}^{t_2} (t_2 - s)\epsilon_n ds \right| \\
 & \quad + \left| \int_{t_0}^{t_2} (t_2 - s)\epsilon_n ds - \int_{t_0}^{t_2} (t_2 - s)\epsilon_n ds \right| \\
 & \leq |f(t_1, r(t_1, \epsilon_n)) - f(t_2, r(t_2, \epsilon_n))| \\
 & \quad + \left| \int_{t_0}^{t_1} (t_1 - t_2)g(s, r_{\epsilon_n}(s)) ds \right| + \left| \int_{t_1}^{t_2} (t_2 - s)g(s, r_{\epsilon_n}(s)) ds \right| \\
 & \quad + \left| \int_{t_0}^{t_1} (t_1 - t_2)\epsilon_n ds \right| + \left| \int_{t_1}^{t_2} (t_2 - s)\epsilon_n ds \right| \\
 & \leq |f(t_1, r(t_1, \epsilon_n)) - f(t_2, r(t_2, \epsilon_n))| \\
 & \quad + \int_{t_0}^{t_0+a} |t_1 - t_2| |g(s, r_{\epsilon_n}(s))| ds + \left| \int_{t_1}^{t_2} |g(s, r_{\epsilon_n}(s))| ds \right| \\
 & \quad + \int_{t_0}^{t_0+a} |t_1 - t_2| \epsilon_n ds + \left| \int_{t_1}^{t_2} (t_0 + a)\epsilon_n ds \right| \\
 & \leq |f(t_1, r(t_1, \epsilon_n)) - f(t_2, r(t_2, \epsilon_n))| \\
 & \quad + a \|h\| |t_1 - t_2| + \left| \int_{t_1}^{t_2} h(s) ds \right| \\
 & \quad + a |t_1 - t_2| \epsilon_n + |t_1 - t_2| (t_0 + a) \epsilon_n \\
 & \leq |f(t_1, r(t_1, \epsilon_n)) - f(t_2, r(t_2, \epsilon_n))| \\
 & \quad + c |t_1 - t_2| + |p(t_1) - p(t_2)|
 \end{aligned}$$

where,  $c = \int_{t_0}^{t_0+a} |h(s)| ds$  and  $c = [(t_0 + 2a)\epsilon_n + a \|h\|]$ .

Since  $f$  is continuous with the compact set  $J \times [-N, N]$ , it is continuous uniformly there.

Hence,

$$|f(t_1, r(t_1, \epsilon_n)) - f(t_2, r(t_2, \epsilon_n))| \rightarrow 0 \text{ as } t_1 \rightarrow t_2$$

Uniformly for all  $n \in \mathbb{N}$ . Similarly, since the function  $p$  is continuous on compact set  $J$ . It is uniformly continuous and hence.

$$|p(t_1) - p(t_2)| \rightarrow 0 \text{ as } t_1 \rightarrow t_2$$

Uniformly for all

$$t_1, t_2 \in J.$$

Therefore, from the above inequality (5.4.5), it follows that

$$|r(t_1, \epsilon_n) - r(t_2, \epsilon_n)| \rightarrow 0 \text{ as } t_1 \rightarrow t_2$$

Uniformly for all  $n \in \mathbb{N}$ . Therefore,

$$r(t, \epsilon_n) \rightarrow r(t) \text{ as } n \rightarrow \infty$$

For all  $t \in J$ . Next, we show that the function  $r(t)$  is a solution of the HIDE (5.3.2)

defined on  $J$ . Now, since  $r(t, \epsilon_n)$  is a solution of the HIDE (5.4.5), we have

$$r(t, \epsilon_n) = x_0 + h(t_0, x_0) + \int_{t_0}^t (t-s)g(s, r_{\epsilon_n}(s)) ds$$

for all  $t \in J$ . Taking the limit as  $n \rightarrow \infty$  in the above equation (5.4.6) yields

$$r(t) = x_0 + h(t_0, x_0) + \int_{t_0}^t (t-s)g(s, r(s)) ds$$

for  $t \in J$ . The function  $r$  is hence a HIDE solution on  $J$ . Lastly, the inequality which follows

$$u(t) \leq r(t)$$

for all  $t \in J$ . Therefore the answer on  $J$  is maximum. This is the evidence.

### COMPARISON THEOREMS

An evaluation of the solution to HIDE-related integrated differential inequality is the main problem of integral inequalities. In this section, we demonstrate that maximum and minimum solutions are within the boundaries of integral inequality

using HIDE solutions on  $J = [t_0, t_0 + a]$ .

**Theorem 1.4** Assume that the hypotheses (A<sub>0</sub>) through (A<sub>2</sub>) hold. Further, if there exists a function  $u \in C(J, \mathbb{R})$  such that

$$\left. \begin{aligned}
 \frac{d}{dt} [u(t) - f(t, u(t))] &\leq \int_{t_0}^t g(s, u(s)) ds \quad t \in J \\
 u(t_0) &\leq x_0.
 \end{aligned} \right\}$$

Then,

$$u(t) \leq r(t)$$

for all  $t \in J$ . Where  $r$  is  $J$ 's highest HIDE solution.

**PROOF.** Let  $\epsilon > 0$  be small and random. The maximum solution is the HIDE by theorem

$$r(t) = \lim_{\epsilon \rightarrow 0} r(t, \epsilon)$$

Is consistent on  $J$ , and the  $r$  function is the highest possible HIDE solution on  $J$ . Therefore, we get

$$\left. \begin{aligned} \frac{d}{dt} [r(t, \epsilon) - f(t, r(t, \epsilon))] &= \int_{t_0}^t g(s, r(s, \epsilon)) + \epsilon, \quad t \in J \\ r(t_0, \epsilon) &= x_0 + \epsilon. \end{aligned} \right\}$$

From the above inequality, it follows that

$$\left. \begin{aligned} \frac{d}{dt} [r(t, \epsilon) - f(t, r(t, \epsilon))] &> \int_{t_0}^t g(s, r(s, \epsilon)), \quad t \in J \\ r(t_0, \epsilon) &> x_0. \end{aligned} \right\}$$

Now we apply Theorem to the inequalities and conclude that  $u(t) < r(t, \epsilon)$  for all  $t \in J$ . This further because of limit implies that inequality holds on  $J$ . This completes the proof.

**Theorem 1.5** Assume that the hypotheses (A<sub>0</sub>) through (A<sub>2</sub>) hold. Further, if there exists a function  $v \in C(J, \mathbb{R})$  such that

$$\left. \begin{aligned} \frac{d}{dt} [v(t) - f(t, v(t))] &\geq \int_{t_0}^t g(s, v(s)), \quad t \in J \\ v(t_0) &\geq x_0. \end{aligned} \right\}$$

Then,

$$\rho(t) \leq v(t)$$

for all  $t \in J$ , where  $\rho$  is a minimal solution of the HIDE on  $J$ .

Note that Theorem is useful to prove the boundedness and uniqueness of the solutions for the HIDE on  $J$ . A result in this direction is

## CONCLUSION

The hybrid integrodifferential equations are a rich area for a variety of nonlinear ordinary as well as partial integrodifferential equations. Here, in this chapter, we have considered a very simple hybrid integro-differential equation involving two non linearities, however, a more complex hybrid integro-differential equation can also be studied on similar lines with appropriate modifications. Again, the results proved in this chapter are very fundamental in nature and therefore, all other problems for the hybrid integro-differential equation in question are still open. In a forthcoming paper we plan to prove some theoretical approximation results for the hybrid integro-differential equation.

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