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A Review of Generalized Fractional Integral Operator in Fractional Differential Equations

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Abstract - As a result of its wide range of applications in science during the past several years, academics have focused a great deal of attention on fractional calculus (FC). Multivariate Mittag–Leffler functions are considered strong extensions of the traditional Mittag–Leffler functions in fractional calculus. An integral operator with a multivariate Mittag–Leffler (M-L) function is introduced in this study. For example, we show that an infinite series of Riemann–Liouville integrals can be expanded, the Laplace transform (LT), semigroup property can be shown, and composition with Riemann–Liouville integrals can be proved for the proposed operators. Also, we discuss the features of fractional differential operators. The suggested operators like the fractional kinetic differential and the time-fractional heat equation are also explored.

Keywords - Fractional Integral Operator, Differential Equations, K_4

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INTRODUCTION

A new contemplate fractional integral operator employing K4 mapping is the focus of this chapter. For the suggested generalized operator, the Mellin and Laplace transforms as well as the research of boundedness and the development of novel composite qualities are also addressed. K4 mapping and Hilfer differentiations are utilized to decipher the fractional derivative equation based on the results reached. Using the K4 mapping as a generalization of the M-series, we may conclude that various outcomes described before are gladly followed as specific findings in our inquiry. Recent research in applied sciences, engineering, and technology has discovered new uses for generalized special functions. These conclusions must be "validated" by the establishment of several corollaries and lemmas.

Several fractional integral and differential operators have been investigated in detail due to the widespread use of fractional calculus by researchers including Kilbas, Kalla, Khan, McBride, Kumar and Saxena, and Kiryakova, amongst others. It was Sharma who first created the K4 function.:

$$K_4^{(\alpha,\beta,\gamma)(c,0)(p,q)}(a_1,\ldots a_p;\ b_1\ldots b_q : x)$$

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n(c)^n}{(n)!}\frac{x^{-\beta+(n+\gamma)\alpha-1}}{\Gamma((n+\gamma)\alpha-\beta)}, \qquad 1$$

where $x,\alpha,\beta\in C$, $\Re(\alpha\gamma-\beta)>0$; $(a_i)_n(i=1,2,\ldots p)$ and $(b_j)_n\ (j=$

are the Pochhammer notation. The above equation (1) is only valid when none one of the variables *j b* is zero or integer (negative) and if any num. variable *i a* is zero or negative integer, the series changes to a polynomial in variable *x*. The series is convergent if *q* +1< *p* .

If we replace , $\beta=\alpha-\beta_1,\ \gamma=1,\ c=1$ in (1), we obtain the subsequent result:

$$K_4^{(\alpha,\alpha-\beta_1,1)(1,0)(p,q)}(a_1,\ldots a_p;b_1,\ldots b_q : x)$$

$$=x^{\beta_1-1}\ {}_p^{\alpha}M_q^{\beta_1}(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots b_q;x^{\alpha}), \qquad 2$$

where ${}_p^{\alpha}M_q^{\beta_1}$ is the well known generalized M-series which was explored by Sharma and

Jain, is denoted as power series:

$${}_p^{\alpha}M_q^{\beta_1}(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots b_q;z)={}_p^{\alpha}M_q^{\beta_1}(z)={}_p^{\alpha}M_q^{\beta_1}\left((a_i)_1^p;(b_j)_1^q:z\right)$$

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{z^n}{\Gamma(\alpha n+\beta_1)},\ (z,\alpha,\beta_1\in C,\ \Re(\alpha)>0), \qquad 3$$

where, $(a_i)_n, (b_j)_n$ are the recognized Pochhammer symbols. The series given in (3) is described as only when none of the variable $b_j's,\ j = 1, 2, \ldots, q$, is zero or a negative integer;

if any numerator variable *i a* is zero or a negative integer, then the series changes to a polynomial in z. If $p \le q$, The equation (3) is confluent for all variable z, if *p* = *q* +1, it is concurrent for $|z| < \delta = \alpha^{\alpha}$ and if *p* > *q* +1, it is divergent. When *p* = *q* + *1* and $|z| = \delta$, equation will converge on conditions based on varibles values.

Prabhakar introduced the generalized Mittag-Leffler mapping which may be found from

(3) for ; $a_1 = \gamma \in C;\ p = 1 = q;\ \ b_1 = 1$ as

$$E^{\gamma}_{\alpha,\beta_1}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{\Gamma(m\alpha+\beta_1)} \frac{z^m}{m!} = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{(1)_m} \frac{z^m}{\Gamma(m\alpha+\beta_1)} = {}^{\alpha}_{1}M^{\beta_1}_{1}(\gamma\,;1\,;z) \quad 4$$

The induced M-series represent by (3) can be revealed as a particular case of Wright

generalized hypergeometric function and Fox H-function as

$${}^{\alpha}_{p}M^{\beta_1}_{q}\left((a_i)^p_1 ; (b_j)^q_1 ; z\right) = k\ {}_{p+1}\psi_{q+1}\left[\begin{matrix}(a_1,1),\ldots,(a_p,1),(1,1); \\ (b_1,1),\ldots,(b_q,1),(\beta_1,\alpha);\end{matrix}\ z\right]$$

$$= kH^{1,p+1}_{p+1,q+2}\left[-z\left|\begin{matrix}(1-a_i,1)^p_1,(0,1) \\ (0,1),(1-b_j,1)^q_1,(1-\beta_1,\alpha)\end{matrix}\right.\right], \quad 5$$

where $k = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$ and $H^{m,n}_{p,q}(z)$ is H function discussed by Fox [6], is presented by Mellin-Barnes type integral such that *m, n, p, q* are integers satisfying $0 \le n \le p;\ 0 \le m \le q$, for $a_i, b_j \in C$ and for $\alpha_i, \beta_j \in \Re_+ = (0,\infty)$, $(i = 1, 2, \ldots p;\ j = 1, 2, \ldots, q)$ as

$$H^{m,n}_{p,q}(z) \equiv H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_i,\alpha_i)^p_1 \\ (b_j,\beta_j)^q_1\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L H^{m,n}_{p,q}(s) z^{-s} ds, \quad 6$$

$$H^{m,n}_{p,q}(s) \equiv H^{m,n}_{p,q}\left[\left.\begin{matrix}(a_i,\alpha_i)^p_1 \\ (b_j,\beta_j)^q_1\end{matrix}\right|\ s\right] = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}, \quad 7$$

together with the convergence conditions as known by Mathai, Braaksma, Kilbas and Saigo, Ram J. and D. Kumar. Wright given generalized hypergeometric function by resources of the sequence representated in the type given as

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p) \\ (b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\ \middle|\ z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+nA_i)z^n}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)n!}, \quad 8$$

where $A_i, B_j \in \Re_+,\ z, a_i, b_j \in C,\ A_i \ne 0,\ B_j \ne 0;\ j = 1,\ldots\ldots,q,\ i = 1,\ldots\ldots,p.$

Hilfer gave fractional differential operator with two variables of order $0 < \mu_1 < 1, 0 < v < 1$ in the form as mantion below:

$$\left(D^{\mu_1,v}_{a+}N(x,t)\right) = \left(I^{(v)(1-\mu_1)}_{a+}\frac{\partial}{\partial x}\left(I^{(1-v)(1-\mu_1)}_{a+}N(x,t)\right)\right). \quad 9$$

The preliminary value term $I^{(1-v)(1-\mu_1)}_{a+}f(0+)$ includes the fractional integral (R-L) of order $(1-v)(1-\mu_1)$.

Lemma:

If $\alpha, \beta, \gamma, z \in C,\ \Re(\alpha\gamma-\beta) > 0,\ \Re(\gamma) > 0, \Re(z) < 0; (a_i)_n\ \ (i = 1, 2, \ldots p)$ and

$(b_j)_n\ \ (j = 1, 2, \ldots q)$ are the Pochhamer notations, the functions represented by (1) can easily be expressed by the Mellin-Barnes types integral as given below:

$$K^{(\alpha,\beta,\gamma)(c,0)(p,q)}_{4}(a_1,\ldots a_p ; b_1,\ldots b_q ; z)$$

$$= \frac{k}{2\pi i\Gamma(\gamma)}\int_{-i\infty}^{i\infty}\frac{\Gamma(\xi)\Gamma(\gamma-\xi)\prod_{i=1}^{p}\Gamma(a_i-\xi)}{\prod_{j=1}^{q}\Gamma(b_j-\xi)\Gamma((\gamma\alpha-\beta)-\alpha\xi)}(-c)^{-\xi}(z)^{((\gamma\alpha-\beta-1)-\alpha\xi)}d\xi, \quad 10$$

where $k = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$, $|\arg(z)| < \pi$ and figure of integral starts at $-i\infty$ and end at $+i\infty$ and separate the poles at the point $\xi = -n (n \in N_0)$ to the left and all the poles of $\Gamma(\gamma-\xi)$ at the point $\xi = \gamma + n (n \in N_0)$ and the poles of $\Gamma(\gamma\alpha-\beta-\alpha\xi)$ at the point $\xi = \frac{\gamma\alpha-\beta}{\alpha}+n$ are put to the right.uu

The goal of this section is to find different qualities like Mellin and Laplace transform of the suggested extended integral operator with function written as follows:

$$\left(K^{(\alpha,\beta,\gamma)(c,0)(p,q)(a_1,\ldots a_p)}_{4b_1,\ldots b_q;\omega,\mu,\delta,a+}\phi\right)(x) = \int_a^x (x-t)^{\mu-1} K^{(\alpha,\beta,\gamma)(c,0)(p,q)(a_1,\ldots a_p)}_{4(b_1,\ldots,b_q)}\left[\omega(x-t)^{\delta}\right]\phi(t)dt, \quad 11$$

where $\alpha, \gamma, \beta, \delta, \mu,\ \omega, x \in C$ and $\Re(\alpha\gamma-\beta) > 0, \Re(\mu) > 0, \Re(\delta) > 0.$ .The operator was studied by Shukhla and Prajapati. It includes special cases.

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$$x^{\beta_1-1}\left({}_p^{\alpha}M_{q(a_1,\ldots,a_p,b_1,\ldots,b_q;\omega,a+}^{\beta_2}\phi\right)(x)=\omega^{\beta_1-1}\int_a^x(x-t)^{\mu+\delta\beta_1-\delta-1}\ {}_p^{\alpha}M_q^{\beta_1}\left(\omega^{\alpha}(x-t)^{\delta\alpha}\right)\phi(t)dt, \quad 12$$

where $\Re(\mu+\delta\beta_1-\delta)>0,\ \Re(\delta\alpha)>0$.

On setting $\beta_1=1,\ a_1=b_1,\ p=q=1,$ equation (12) it capitulates the following operator

linked with Mittag Leffler function studied by Kiryakova.

$$\left(E_{\alpha;\omega,\mu,\delta,a+}\phi\right)(x)=\int_a^x(x-t)^{\mu-1}E_{\alpha}\left[\omega^{\alpha}(x-t)^{\delta\alpha}\right]\phi(t)dt, \quad 13$$

where $\Re(\mu)>0, \Re(\delta\alpha)>0,\ c,\ \alpha,\ \omega\in C,\ x>a.$

MELLIN TRANSFORM OF THE PROPOSED K_4 INTEGRAL OPERATOR

Theorem:

If $\alpha,\beta,\gamma,x,\omega\in C,\ \Re(\alpha\gamma-\beta)>0,\quad \Re(\mu)>0,\ \Re(\delta)>0,\ \Re(\gamma)>0,$

$\Re(1-s-\mu-\delta(\alpha\gamma-\beta-1))>0,\ \Re(a_i)>0,\ \Re(b_j)>0$ and $\Re(\mu+\delta(\gamma\alpha-\beta-1))>0,\ x>0$ the following conclusions holds true:

$$M\left[\left(K_{4_{(b_1,\ldots,b_q),\omega,\mu,\delta,0+}}^{(\alpha,\beta,\gamma),(c,0);(p,q)(a_1,\ldots,a_p)}\phi\right)(x);s\right]$$

$$=\frac{k\omega^{\gamma\alpha-\beta-1}}{\Gamma(\gamma)\Gamma(1-s)}H_{p+2,q+3}^{2,p+2}\left[-c\omega^{\alpha}u^{\delta\alpha}\left|\begin{matrix}(1-\alpha_i,1)_{1,p}(1-\gamma,1)(1-\mu-\delta(\gamma\alpha-\beta-1),\delta\alpha)\\(0,1)(1-s-\mu-\delta(\gamma\alpha-\beta-1),\delta\alpha)(1-b_j,1)_{1,q}(1-\gamma\alpha+\beta,\alpha)\end{matrix}\right.\right].\ M\left(t^{\mu+\delta(\gamma\alpha-\beta-1)}\phi(t)\right), \quad 14$$

with $k=\dfrac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$ and $H_{p,q}^{m,n}(\bullet)$ is the Fox H-Function explored by Fox

Proof: The Mellin transform as defined follows:

$$M(\phi(x);s)=\int_{x=0}^{\infty}x^{s-1}\phi(x)dx$$

and using eq.(11), we obtain left hand side of (14)

$$M\left[\left(K_{4_{(b_1,\ldots,b_q),\omega,\mu,\delta,0+}}^{(\alpha,\beta,\gamma),(c,0);(p,q)(a_1,\ldots,a_p)}\phi\right)(x);s\right]=\int_{x=0}^{\infty}x^{s-1}\left[\int_{t=0}^{x}(x-t)^{\mu-1}K_4\left(\omega(x-t)^{\delta}\right)\phi(t)dt\right]dx. \quad 15$$

By replacing the order of integral, as admissible under the same conditions stated in theorem

$$M\left[\left(K_{4_{(b_1,\ldots,b_q),\omega,\mu,\delta,0+}}^{(\alpha,\beta,\gamma),(c,0);(p,q)(a_1,\ldots,a_p)}\phi\right)(x);s\right]=\int_{t=0}^{\infty}\phi(t)\int_{x=t}^{\infty}x^{s-1}(x-t)^{\mu-1}K_4\left[\omega(x-t)^{\delta}\right]dx\,dt, \quad 16$$

If we replace $x=t+u$ on the R.H.S. of the equation (4.16), we obtain

$$M\left[\left(K_{4_{(b_1,\ldots,b_q),\omega,\mu,\delta,0+}}^{(\alpha,\beta,\gamma),(c,0);(p,q)(a_1,\ldots,a_p)}\phi\right)(x);s\right]=\int_{t=0}^{\infty}\phi(t)\int_{u=0}^{\infty}(t+u)^{s-1}u^{\mu-1}K_4\left[\omega u^{\delta}\right]du\,dt.$$

At the present on indicating *K4* function in reference of its Mellin-Bernes contour integration of equation (10), we see

$$M\left[\left(K_{4_{(b_1,\ldots,b_q),\omega,\mu,\delta,0+}}^{(\alpha,\beta,\gamma),(c,0);(p,q)(a_1,\ldots,a_p)}\phi\right)(x);s\right]$$

$$=\int_{t=0}^{\infty}\phi(t)dt.\frac{k}{2\pi i\Gamma(\gamma)}\int_{-i\infty}^{i\infty}\frac{\Gamma(\xi)\Gamma(\gamma-\xi)\prod_{i=1}^{p}\Gamma(a_i-\xi)}{\prod_{j=1}^{q}\Gamma(b_j-\xi)\Gamma((\gamma\alpha-\beta)-\alpha\xi)}(-c)^{-\xi}(\omega)^{((\gamma\alpha-\beta-1)-\alpha\xi)}d\xi$$

$$.\int_{u=0}^{\infty}(t+u)^{s-1}u^{\mu+\delta(-\xi\alpha+\gamma\alpha-\beta-1)-1}du. \quad 17$$

While using following formula to solve u’s integral at right hand side in equation (17)

$$\int_{t=0}^{\infty}(t+a)^{-\beta}t^{\tau-1}dt=\frac{\Gamma(\tau)\Gamma(\beta-\tau)}{\Gamma(\beta)}a^{\tau-\beta},$$

with $\Re(\tau)>0,\ \Re(\beta-\tau)>|\arg(-a)|<\pi..$

Now by goodness of definition of H function in terms of Mellin-Bernes contour integration(6, 7) resist the desired results (14).

Corollary:

On setting $\beta=\alpha-\beta_1,\ \gamma=1,\ c=1$ equation (14) yields the following result

$$M\left[x^{\beta_1-1}\ {}_p^{\alpha}M_{q;\omega,\mu,\delta,a+}^{\beta_1}\left((a_i)_1^p;(b_j)_1^q;\phi\right)(x^{\alpha})\right]$$

$$=\frac{k\omega^{\beta_1-1}}{\Gamma(1-s)}H_{p+2,q+3}^{2,\ p+2}\left[-\omega^{\alpha}u^{\delta\alpha}\left|\begin{matrix}(1-a_j,1)_{1,p}(0,1)(1-\mu-\delta(\beta_1-1),\delta\alpha)\\(0,1)(1-s-\mu-\delta(\beta_1-1),\delta\alpha)(1-b_j,1)_{1,q}(1-\beta_1\alpha)\end{matrix}\right.\right]M\left(t^{\mu+\delta(\beta_1-1)}\phi(t)\right) \quad 18$$

where $k=\dfrac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$.

LAPLACE TRANSFORM OF PROPOSED K_4 OPERATOR

Theorem

If $\Re(\mu)>0,\ \Re(s)>0,\ \Re(\delta)>0\ \ \Re(\gamma)>0, \alpha, \beta, \gamma, \omega \in C, \Re(\alpha\gamma-\beta)>0$, we

$$L\left[\left(K_{4\left(b_1,\dots,b_q\right);\,\omega,\,\mu,\,\delta,\,0+}^{(\alpha,\beta,\gamma),(c,0):(p,q)\left(a_1\dots a_p\right)}\phi\right)(x);s\right]$$

$$=\frac{k}{\Gamma(\gamma)}s^{-\mu}\left(\frac{\omega}{s^{\delta}}\right)^{\gamma\alpha-\beta-1}{}_{p+2}\psi_{q+1}\left[\begin{matrix}\left(a_j,1\right)_{1,p}(\gamma,1)(\mu+\delta(\gamma\alpha-\beta-1),\delta\alpha)\\ \left(b_j,1\right)_{1,q}(\gamma\alpha-\beta,\alpha)\end{matrix};c\left(\frac{\omega}{s^{\delta}}\right)^{\alpha}\right]\tilde{\phi}(s). \qquad 19$$

where $k=\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma\left(a_i\right)}$ and ${}_p\psi_q(\bullet)$ is Wright function as given by eq.(8).

Proof: By applying the result (11), we obtain

$$L\left[\left(K_{4\left(b_1,\dots,b_q\right);\,\omega,\,\mu,\,\delta,\,0+}^{(\alpha,\beta,\gamma),(c,0):(p,q)\left(a_1\dots a_p\right)}\phi\right)(x);s\right]=\int_{x=0}^{\infty}e^{-sx}\left[\int_{t=0}^{x}(x-t)^{\mu-1}K_4\left[\omega(x-t)^{\delta}\right]\phi(t)dt\right]dx,$$

by replacing the order of integration, admissible under the same circumstances declared in theorem 19

$$L\left[\left(K_{4\left(b_1,\dots,b_q\right);\,\omega,\,\mu,\,\delta,\,0+}^{(\alpha,\beta,\gamma),(c,0):(p,q)\left(a_1\dots a_p\right)}\phi\right)(x);s\right]=\int_{t=0}^{\infty}\phi(t)\int_{x=t}^{\infty}e^{-sx}(x-t)^{\mu-1}K_4\left[\omega(x-t)^{\delta}\right]dx\,dt. \qquad 20$$

If we replace $x = t + u$ on the R.H.S. of the equation (20), we obtain

$$L\left[\left(K_{4\left(b_1,\dots,b_q\right);\,\omega,\,\mu,\,\delta,\,0+}^{(\alpha,\beta,\gamma),(c,0):(p,q)\left(a_1\dots a_p\right)}\phi\right)(x);s\right]=\int_{t=0}^{\infty}\phi(t)\int_{u=0}^{\infty}e^{-s(t+u)}u^{\mu-1}K_4\left[\omega u^{\delta}\right]du\,dt,$$

while using series definition of K_4 function (1), we see

$$L\left[\left(K_{4\left(b_1,\dots,b_q\right);\,\omega,\,\mu,\,\delta,\,0+}^{(\alpha,\beta,\gamma),(c,0):(p,q)\left(a_1\dots a_p\right)}\phi\right)(x);s\right]$$

$$=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\dots\left(a_p\right)_n}{\left(b_1\right)_n\dots\left(b_q\right)_n}\frac{(\gamma)_n c^n}{n!}\frac{\omega^{\alpha n+\gamma\alpha-\beta-1}}{\Gamma((\gamma\alpha-\beta)+n\alpha)}\int_{t=0}^{\infty}\phi(t)e^{-st}dt.\int_{u=0}^{\infty}e^{-su}u^{\mu-1+\delta(\alpha n+\gamma\alpha-\beta-1)}du$$

$$=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\dots\left(a_p\right)_n}{\left(b_1\right)_n\dots\left(b_q\right)_n}\frac{(\gamma)_n c^n}{n!}\frac{\omega^{\alpha n+\gamma\alpha-\beta-1}}{\Gamma((\gamma\alpha-\beta)+n\alpha)}\frac{\Gamma(\mu+\delta(\alpha n+\gamma\alpha-1-\beta))}{s^{(\mu+\delta(\alpha n+\gamma\alpha-1-\beta))}}.\int_{0}^{\infty}\phi(t)e^{-st}dt$$

$$=\frac{k\omega^{\gamma\alpha-1-\beta}}{\Gamma(\gamma)s^{(\mu+\delta(\gamma\alpha-\beta-1))}}{}_{p+2}\psi_{q+1}\left[\begin{matrix}\left(a_j,1\right)_{1,p}(\gamma,1)(\mu+\delta(\gamma\alpha-\beta-1)\delta\alpha)\\ \left(b_j,1\right)_{1,q}(\gamma\alpha-\beta,\alpha)\end{matrix};\ c\left(\frac{\omega}{s^{\delta}}\right)^{\alpha}\right].\tilde{\phi}(s).$$

which completes the proof of theorem where $\tilde{\phi}(S)$ is represent the Laplace transform of $\phi(t)$; and k has same meaning as defined in (19).

K_4 OPERATOR OF POWER FUNCTION

Theorem:

If the conditions mentioned in the equation (4.11) is satisfied and $\Re(\rho)>0, x>a$ then

$$K_{4\left(b_1,\dots,b_q\right);\,\omega,\,\mu,\,\delta,\,a+}^{(\alpha,\beta,\gamma),(c,0):(p,q)\left(a_1\dots,a_p\right)}\left[(t-a)^{\rho-1}\right](x)=\frac{\Gamma(\rho)}{\Gamma(\gamma)}\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{j=1}^{p}\Gamma\left(a_j\right)}\omega^{\gamma\alpha-\beta-1}.(x-a)^{\mu+\delta(\gamma\alpha-\beta-1)+\rho-1}$$

$$\cdot{}_{p+2}\psi_{q+2}\left[\begin{matrix}\left(a_j,1\right)_{1,p}(\gamma,1)(\mu+\delta(\gamma\alpha-\beta-1),\delta\alpha)\\ \left(b_j,1\right)_{1,q}(\gamma\alpha-\beta,\alpha)(\mu+\delta(\gamma\alpha-\beta-1)+\rho,\delta\alpha)\end{matrix};c\omega^{\alpha}(x-a)^{\delta\alpha}\right] \qquad 21$$

Proof: On making use of equation (1) and (11), we see L.H.S. of equation (22)

$$=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\dots\left(a_p\right)_n}{\left(b_1\right)_n\dots\left(b_q\right)_n}\frac{(\gamma)_n c^n}{n!}\frac{\omega^{\alpha n+\gamma\alpha-\beta-1}}{\Gamma((\gamma\alpha-\beta)+n\alpha)}\int_{a}^{x}(x-t)^{\mu-1+\delta(\alpha n+\gamma\alpha-\beta-1)}(t-a)^{\rho-1}dt,$$

after simplification of integral, we get

$$=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\dots\left(a_p\right)_n}{\left(b_1\right)_n\dots\left(b_q\right)_n}\frac{(\gamma)_n c^n}{n!}\frac{\omega^{\alpha n+\gamma\alpha-\beta-1}}{\Gamma((\gamma\alpha-\beta)+n\alpha)}(x-a)^{\mu+\delta(n\alpha+\alpha\gamma-1-\beta)+\rho-1}$$

$$.\frac{\Gamma(\mu+\delta(n\alpha+\alpha\gamma-1-\beta))\Gamma(\rho)}{\Gamma(\mu+\delta(n\alpha+\alpha\gamma-1-\beta)+\rho)}. \qquad 22$$

While taking use of series definition of Wright function (8) yields the result (22) after simple modification.

BOUNDEDNESS OF THE K_4 OPERATOR

Theorem:

With all constraints as stated in equation (11), and the mapping g be the *nth* space $L(a,b)$ of the Lebesgue measurable mappings on limited interval $[a,b]$ $(a < b)$ of the real line $\Re$ is defined by

$$L(a,b)=\left\{g(x):\|g_1\|_1\right\}=\int_{a}^{b}|g(x)|dx<\infty. \qquad 23$$

The integral defined by (11) is given as

$$\left\|K_{4\cdot\left(b_1,\dots b_q\right);\,\omega,\,\mu,\,\delta,\,a+}^{(\alpha,\beta,\gamma);(c,0);(p,q);\left(a_1,\dots,a_p\right)}\phi\right\|\le B\|\phi\|_1, \qquad 24$$

where B is given by

$$B=(b-a)^{\delta(\gamma\alpha-\beta-1)+\mathrm{Re}(\mu)}$$

$$.\sum_{n=0}^{\infty}\frac{\left|a_1\right|\dots\left|a_p\right|_n}{\left|b_1\right|\dots\left|b_q\right|_n}\frac{\left|(\gamma)_n c^n\right|}{n!}\frac{\left|\left(\omega^{\alpha n+\gamma\alpha-\beta-1}\right)(b-a)^{\delta\alpha n}\right|}{\Gamma(\gamma\alpha+n\alpha-\beta)}\frac{1}{\left\|\left[\delta(n\alpha+\gamma\alpha-1-\beta)+\Re(\gamma)\right]\right\|}.$$

Proof: By using of (24) in L.H.S. of (25), we obtain

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If we change the order of integration that is allowable below the same circumstances declared earlier, we get

$$\left\| K_4 .\begin{matrix} (\alpha,\beta,\gamma);(c,0);(p,q);(a_1,\ldots a_p) \\ (b_1,\ldots b_q);\omega,\mu,\delta,a+ \end{matrix} \phi \right\|$$

$$\le \int_{t=a}^{b}\left[\int_{x=t}^{b}(x-t)^{\mathrm{Re}(\mu)-1}\left|K_{4}{}^{(\alpha,\beta,\gamma),(c,0)(p,q)(a_1,\ldots,a_p)}_{(b_1,\ldots,b_q)};\omega(x-t)^{\delta}\right|dx\right]|\phi(t)|dt.$$

Put $x=t+u$, we get

$$\left\| K_4 .\begin{matrix} (\alpha,\beta,\gamma);(c,0);(p,q);(a_1,\ldots a_p) \\ (b_1,\ldots b_q);\omega,\mu,\delta,a+ \end{matrix} \phi \right\| \le \int_{t=a}^{b}\left[\int_{u=0}^{b-t}u^{\Re(\mu)-1}\left|K_4;\omega u^{\delta}\right|du\right]|\phi(t)dt|.$$

On using series expressed as K_4 function (1), we have

$$\left\| K_4 .\begin{matrix} (\alpha,\beta,\gamma);(c,0);(p,q);(a_1,\ldots a_p) \\ (b_1,\ldots b_q);\omega,\mu,\delta,a+ \end{matrix} \phi \right\|$$

$$\le \int_{t=a}^{b}\left[\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n c^n}{n!}\frac{\omega^{\alpha n+\gamma\alpha-\beta-1}}{\Gamma((\gamma\alpha-\beta)+n\alpha)}\int_{u=0}^{b-t}u^{(\alpha n+\gamma\alpha-\beta-1)\delta+\Re(\mu)-1}du\right]|\phi(t)|dt$$

$$=\int_{t=a}^{b}\left[\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n c^n}{n!}\frac{\omega^{\alpha n+\gamma\alpha-\beta-1}}{\Gamma((\gamma\alpha-\beta)+n\alpha)}\frac{(b-t)^{(\alpha n+\gamma\alpha-\beta-1)\delta+\Re(\mu)}}{((\alpha n+\gamma\alpha-\beta-1)\delta+\Re(\mu))}\right]|\phi(t)|dt,$$

$$\left\| K_4 \begin{matrix} (\alpha,\beta,\gamma);(c,0);(p,q);(a_1,\ldots a_p) \\ (b_1,\ldots b_q);\omega,\mu,\delta,a+ \end{matrix} \phi \right\| \le (b-a)^{(\gamma\alpha-\beta-1)\delta+\Re(\mu)}$$

$$.\sum\frac{|a_1|..|a_p|_n}{|b_1|_n..|b_q|_n}\frac{|(\gamma)_n c^n|}{n!}\frac{|(\omega^{\alpha n+\gamma\alpha-1-\beta})(b-a)^{\delta\alpha n}|}{\Gamma(\gamma\alpha+\alpha n-\beta)}.\frac{1}{[\delta(\alpha n+\gamma\alpha-1-\beta)+\Re(\mu)]}\int_a^b|\phi(t)|dt=B\|\phi\|_1.$$

This completes the proof of result shown by equation (26).

APPLICATION OF PROPOSED K_4 OPERATOR IN SOLVING FRACTIONAL DERIVATIVE EQUATION

Theorem:

With all limitations on variables as given in equation (19) the solution of the Fractional derivative equation

$$\left(D_{0+}^{\mu_1,\nu}f\right)(x)=\lambda\left(K_{4}{}^{(\alpha,\beta,\gamma),(c,0):(p,q)(a_1,\ldots,a_p)}_{(b_1,\ldots,b_q);\omega,\mu,\delta,0+}\right)(x)+g(x),$$

with initial constraints $\left(I_{0+}^{(1-\nu)(1-\mu_1)}f\right)(0+)=e$

is given by as

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$$f(x)=e\frac{x^{\mu_1-\nu(1-\mu_1)-1}}{\Gamma(\mu_1-\nu(1-\mu_1))}$$

$$+\lambda\frac{k}{\Gamma(\gamma)}(\omega)^{\gamma\alpha-\beta-1}x^{\mu_1+\mu+\delta(\gamma\alpha-\beta-1)}\;{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_j,1)_{1,p}(\gamma,1)(\mu+\delta(\gamma\alpha-\beta-1),\delta\alpha)\\(b_j,1)_{1,q},(\delta(\alpha\gamma-\beta-1)+\mu_1+\mu+1,\alpha\delta)(\gamma\alpha-\beta,\alpha)\end{matrix};c\omega^{\alpha}x^{\delta\alpha}\right]$$

$$+\frac{1}{\Gamma(\mu_1)}\int_0^x(x-t)^{\mu_1-1}g(t)dt,$$

where $0<\mu_1<1,\ 0\le\nu\le1,\ \omega\in C$ and $k=\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$, e is whimsical constant and ${}_p\psi_q(\bullet)$ is expanded Wright function.

Proof: Put in application the Laplace transform to both side of (27) and applying Convolution theorem of the Laplace transform

$$s^{\mu 1}F(s)-es^{\nu(1-\mu_1)}=\lambda L\left[\left(x^{\mu-1}K_4(\omega x^{\delta}):s\right)\right]L[(1,s)]+G(s),$$

$$s^{\mu_1}F(s)-es^{\nu(1-\mu_1)}$$

$$=\lambda\frac{k}{\Gamma(\gamma)}s^{-\mu-1}\left(\frac{\omega}{s^{\delta}}\right)^{\gamma\alpha-\beta-1}\;{}_{p+2}\Psi_{q+1}\left[\begin{matrix}(a_j,1)_{1,p}(\gamma,1)(\mu+\delta(\gamma\alpha-\beta-1),\delta\alpha)\\(b_j,1)_{1,q},(\gamma\alpha-\beta,\alpha)\end{matrix};c\left(\frac{\omega}{s^{\delta}}\right)^{\alpha}\right]+G(s),$$

$$F(s)=es^{\nu(1-\mu_1)-\mu_1}$$

$$+\lambda\frac{k}{\Gamma(\gamma)}(\omega)^{\gamma\alpha-\beta-1}s^{-\delta(\gamma\alpha-\beta-1)-\mu-\mu_1-1}\;{}_{p+2}\Psi_{q+1}\left[\begin{matrix}(a_j,1)_{1,p}(\gamma,1)(\mu+\delta(\gamma\alpha-\beta-1),\delta\alpha)\\(b_j,1)_{1,q},(\gamma\alpha-\beta,\alpha)\end{matrix};c\left(\frac{\omega}{s^{\delta}}\right)^{\alpha}\right]+\frac{G(s)}{s^{\mu_1}}.$$

On imitating inverse Laplace transform and definition of Wright function

$$f(x)=e\frac{x^{\mu_1-\nu(1-\mu_1)-1}}{\Gamma(\mu_1-\nu(1-\mu_1))}$$

$$+\lambda\frac{k}{\Gamma(\gamma)}(\omega)^{\gamma\alpha-\beta-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+n)}{\prod_{j=1}^{q}\Gamma(b_j+n)}\frac{\Gamma(\gamma+n)}{n!}\frac{\Gamma(\mu+\delta(\gamma\alpha-\beta-1)+\alpha\delta n)}{\Gamma(\gamma\alpha-\beta+\alpha n)}$$

$$c^n\omega^{\alpha n}L^{-1}\left[s^{-\delta(\gamma\alpha-\beta-1)-\mu_1-\mu-1-\alpha\delta n};x\right]+L^{-1}\left[L\left(\frac{x^{\mu_1-1}}{\Gamma(\mu_1)}\right).L(g(x))\right].$$

That yields

$$f(x)=e\frac{x^{\mu_1-\nu(1-\mu_1)-1}}{\Gamma(\mu_1-\nu(1-\mu_1))}$$

$$+\lambda\frac{k}{\Gamma(\gamma)}(\omega)^{\gamma\alpha-\beta-1}x^{\mu_1+\mu+\delta(\gamma\alpha-\beta-1)}\;{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_j,1)_{1,p}(\gamma,1)(\mu+\delta(\gamma\alpha-\beta-1),\delta\alpha)\\(b_{j,1})_{1,q},(\delta(\gamma\alpha-\beta-1)+\mu_1+\mu+1,\alpha\delta)(\gamma\alpha-\beta,\alpha)\end{matrix};c\omega^{\alpha}x^{\delta\alpha}\right]$$

$$+\frac{1}{\Gamma(\mu_1)}\int_0^x(x-t)^{\mu_1-1}g(t)dt.$$

Which yields proof of theorem as required.

CONCLUSION

In this chapter, we created a generalized integral operator that incorporates functions as well. In many physical science equations, such as the diffusion equation, fractional kinetic equation, vibrations, etc., numerous abstract findings based on differential operator theory may be deduced from these results. The Mellin and Laplace transforms have been examined using the proposed fractional integral operator. Eventually, these assumptions are utilized to solve fractional derivative equations, including mapping and associated with Hilfer's differentiations. Assumptions are typically made in many ways in the fractional integral operator of particular mappings.

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