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Solution of Dual Integral Equations Involving Generalized Function

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Abstract – This article's main goal is to solve a dual integral equation by lowering it to an integral equation through the use of mellin transform whose kernel includes generalized polynomial function. We assume that there are definitely many ways to reduce these dual integral formulas by using various transformations such as Fourier, Henkel, etc. For the reason of illustration we pick a dual integral equation of particular type and reduced it, by use of fractional operators and Mellin transform, to an integral equation.

Keywords – Generalized Polynomial Function; Mellin Transform; Fractional operators; Fox-H function.

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1. INTRODUCTION

In several fields of mathematical physics, dual integral equations are often found and usually occur when solving a boundary value problem with mixed boundary conditions. The present paper attempts to solve such dual integral equations that involve generalized polynomial function as a kernel by reducing them into integral equations. Many attempts have been made in the past to solve such problems. The following integral equations are basic tool for our illustration.

$$\int_0^\infty k_1(x,u)\ A(u)\,du = \lambda(x);\ 0 \le x \le 1 \qquad (1.1)$$

$$\int_0^\infty k_2(x,u)\ A(u)\,du = \omega(x);\ x > 1 \qquad (1.2)$$

k_1 & k_2 are kernels defind over $x-u$ plane.

$$F_n^r(x,a,k,p) = x^{-a} e^{px^r} D^n \left\{ x^{a+kn} e^{-px^r} \right\} \ a,k,p,r \ \ parameter.$$

2. THEOREM

If f is unknown function satisfying the dual integral equation.

$$\int_0^\infty (x\,|\,y)^{a_1} e^{-(x|y)^r} F_n^r(x\,|\,y;a_1,k,1) f(y)\,dy = h(x);\ 0 \le x < 1 \qquad (2.1)$$

$$\int_0^\infty (x\,|\,y)^{a_2} e^{-(x|y)^r} F_n^r(x\,|\,y;a_2,k,1) f(y)\,dy = g(x);\ \ 1 \le x < \infty \qquad (2.2)$$

When h and g are prescribed function and a_1, a_2 *and* r are parameters, then f is giving by

$$f(x) = \frac{1}{r}\int_0^\infty L(x\,|\,y) t(y)\,dy$$

Where

$$L(x) = H_{2,1}^{1,0}\left(x \left| \begin{matrix} (1,1)\left(\frac{1}{r}(a_1+(k-1)n)\right), \frac{1}{r} \\ ((k-1)n,1) \end{matrix} \right. \right)$$

and

$$t(x) = h(x), 0 \le x < 1$$

$$t(x) = \frac{r\,x^{(k-1)n+a_1}}{\left|\left(\frac{1}{r}(a_2-a_1)\right)\right.} \int_0^\infty (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{-(k-1)n-a_2+r-1} g(v)\,dv; 1 \le x < \infty$$

3. MATHEMATICAL PRELIMINARY

To prove the theorems we shall use Mellin transformer and fractional integral operator.

$$f^*(s) = M[f(x);s] = \int_0^\infty f(x) x^{s-1}\ dx \qquad (3.1)$$

When $s = \sigma + i\tau$ is a complex variable.

The inverse melling transform f(x) of $f^*(s)$ is given by

$$M^{-1}[f^*(s)] = f(x) = \frac{1}{2\pi i}\int_{\sigma+i\infty}^{\sigma+i\infty} f^*(s)\, x^{-s}\, ds \qquad (3.2)$$

Fractional integral operator

$$\tau(\alpha;\beta;r;w(x)) = \frac{r\, x^{-r\alpha+r-\beta-1}}{\Gamma(\alpha)}\int_0^{\infty}(x^r - v^r)^{\alpha-1} v^{\beta} w(v)\, dv \qquad (3.3)$$

$$R(\alpha;\beta;r;w(x)) = \frac{r x^{\beta}}{\Gamma(\alpha)}\int_x^{\infty}(v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} w(v)\, dv \qquad (3.4)$$

4. SOLUTION

Now taking

$$k_i(x) = x^{a_i}\, e^{-x^r} F_n^r(x, a_i, k, 1),\ i = 1,2$$

Then from Erdeeyi [11] We get

$$k_i^*(s) = \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_i)\right)}{r\,\Gamma(s-n)},\ i = 1,2 \qquad (4.1)$$

Hence by use of convolution theorem of Mellin transform,(2.1) & (2.2) can be written as

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_1)\right)}{\Gamma(s-n)} f^*(1-s)\, x^{-s}\, ds = h(x);\ 0 \le x < 1 \qquad (4.2)$$

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_2)\right)}{\Gamma(s-n)} f^*(1-s)\, x^{-s}\, ds = g(x);\ 1 \le x < \infty \qquad (4.3)$$

Now operating a (4.2) by the operator (3.5) we get

$$\frac{rx^{\beta}}{\Gamma\alpha}\cdot\frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_2)\right)}{\Gamma(s-n)} f^*(1-s)\, x^{-s}\, ds \int_x^{\infty}(v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

$$= \frac{rx^{\beta}}{\Gamma\alpha}\int_x^{\infty}(v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

Now putting $v^r = \frac{x^r}{t}$ and simplifying we get

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_2)\right)}{\Gamma(s-n)}\cdot\frac{x^{-s}}{\Gamma\alpha}\int_0^1 (1-t)^{\alpha-1}.t^{\left(\frac{\beta+s}{r}-1\right)} dt\ f^*(1-s)ds \qquad (4.4)$$
$$= \frac{rx^{\beta}}{\Gamma\alpha}\int_x^{\infty}(v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

$$\Rightarrow \frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_2)\right)}{\Gamma(s-n)}\cdot\frac{x^{-s}}{\Gamma\alpha}\frac{\Gamma\left(\frac{1}{r}(\beta+S)\right)\Gamma\alpha}{\Gamma\left(\alpha+\frac{1}{r}\beta+\frac{1}{r}s\right)} f^*(1-s)\,ds = \frac{rx^{\beta}}{\Gamma\alpha}\int_x^{\infty}(v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

$$\Rightarrow \frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_2)\right)}{\Gamma(s-n)} x^{-s}\frac{\Gamma\frac{1}{r}(\beta+s)}{\Gamma\left(\alpha+\frac{1}{r}\beta+\frac{1}{r}s\right)} \times f^*(1-s)\,ds = \frac{rx^{\beta}}{\Gamma\alpha}\int_x^{\infty}(v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

In equation (4.4), we put $\beta = (k-1)n + a_1$ and $\alpha = \frac{1}{r}(a_2 - a_1)$, so that (4.4) Changes to

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_2)\right)}{\Gamma(s-n)} x^{-s}\frac{\Gamma\frac{1}{r}(s+(k-1)n+a_1)}{\Gamma\left(\frac{a_2}{r}-\frac{a_1}{r}+\frac{1}{r}((k-1)n+a_1)+\frac{1}{r}s\right)} f^*(1-s)\,ds = \frac{r x^{(k-1)n+a_1}}{\Gamma\left(\frac{1}{r}(a_2-a_1)\right)}$$

$$\times\int_x^{\infty}(v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{(k-1)n-a_2+r-1} g(v)dv\ \ 1 \le x \le \infty$$

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_1\right)}{\Gamma(s-n)} x^{-s} f^*(1-s)ds = \frac{r x^{(k-1)n+a_1}}{\Gamma\left(\frac{1}{r}(a_2-a_1)\right)}\int_x^{\infty}(v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{(k-1)n-a_2+r-1} g(v)dv\ ;1 \le x \le \infty \qquad (4.5)$$

Now we write

$$t(x) = h(x), \quad 0 \le x < 1$$

and

$$t(x) = \frac{r x^{(k-1)n+a_1}}{\Gamma\left(\frac{1}{r}(a_2-a_1)\right)}\int_x^{\infty}(v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{-(k-1)n-a_2+r-1} g(v)\, dv;1 \le x < \infty$$

(4.6)

Now from (4.2), (4.5), (4.6) we get

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_1\right)}{\Gamma(s-n)} x^{-s}\ f^*(1-s)ds = t(x) \qquad (4.7)$$

Again using convolution theorem of Mellin transform, (4.1) & (4.6) becomes

$$\int_0^{\infty} k_1(x\,|\,y)\, f(y)\, dy = t(x);\ 0 \le x < \infty \qquad (4.8)$$

When

$$k_1(x) = x^{a_1} e^{-x^r} F_n^r(x, a_1, k, 1)$$

Thus pair at dual integral equation (1.1) & (1.2) we have been reduced to single integral equation (4.8). Hence by mellin transform (4.8) can be written as –

$$k_1^*(s)\, f^*(s) = T^*(s) \qquad (4.9)$$

Where

$$k_1^*(s) = \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s+(k-1)n+a_1)\right)}{\Gamma(s-n)}$$

and $T^*(s)$ is the mellin transform of t(x).

$$f^*(s) = L^*(s)\, T^*(s) \qquad (4.10)$$

$$L^*(s) = \frac{1}{k^*(s)} = \frac{\left|\overline{s+(k-1)n}\right.}{\left|\overline{(s)}\right.\left|\overline{\frac{1}{r}(s+(k-1)n+a_1)}\right.}$$

By use of definition of H – function, we get the inverse transform $L(x)$ at $L^*(s)$ as

$$L(x) = H_{2,1}^{1,0}\left(x \left| \begin{matrix} (1,1)\left(\frac{1}{r}(a_1+(k-1)n)\right), \frac{1}{r} \\ ((k-1)n,1) \end{matrix} \right. \right) \qquad (4.11)$$

Taking inverse mellin transform of (4.10)

$$f(x) = \int_0^\infty L(x \mid y)\, t(y)\, dy$$

Hence using (4.11) we get

$$f(x) = \frac{1}{r}\int_0^\infty H_{2,1}^{1,0}\left(x \mid y \left| \begin{matrix} (1,1)\left(\frac{1}{r}(a_1+(k-1)n)\right), \frac{1}{r} \\ ((k-1)n,1) \end{matrix} \right. \right) t(y)dy$$

When t(y) is given by (4.6).

Hence proved the theorem.

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