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Integral Equation Involving Bessel Polynomial Suggested by Hermite Polynomial

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Abstract – The aim of this paper is to find the inversion of such integral equations whose kernel involves many well known classical polynomials like those of Hermite, Laguerre, Bessel, Legendre, Jacobi etc. We believe that many polynomials can be obtained by considering suitable parameters involved in Generalized Hermite Polynomial. For the purpose of illustration we took a Bessel polynomial in form of Generalized Hermite Polynomial by choosing suitable parameters involved in it.

Keywords: Generalized Hermite Polynomial, Mellin Transform, Convolution Theorem, Fox-H function.

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1. INTRODUCTION

Integral equations and their applications constitute a fascinating area of applied mathematics. The Generalized Hermite polynomial is a powerful tool to solve many integral equations. Many boundary value problems reduced to the problem of solving integral equations whose kernel involves many well known classical polynomials like those of Hermite, Laguerre, Bessel, Legendre, Jacobi etc. During the recent past attempts have been made to generalize these classical polynomials with the help of Rodrigue's formulae. To mention Goued Hopper[11] gave a generalization of Hermite polynomials by formulae.

$$H_n^r(x,a,p) = (-1)^n x^{-a} e^{px^r} D^n\left[x^a e^{-px^r}\right] \qquad (1.1)$$

and we have used

$$H_n^{(-1)}(x,a-2,b) = (-1)^n x^{-(a-2)} e^{bx^{-1}} D^n\left[x^{(a-2)} e^{-bx^{-1}}\right] \qquad (1.2)$$

where $D=\frac{d}{dx}$ and $r, a,$ and p are parameters, for suitable value of $r, a,$ and p (1.1) reduced to modified Hermite, modified Laguerre and modified Bessel polynomials. In view of these generalizations it is worth considering integral equations involving $H_n^{(-1)}(x,a-2,b)$ as kernel and such we prove the following theorem.

2. THEOREM

If f is an unknown function satisfying the integral equation.

$$g(x) = \int_0^\infty k(x \mid y) f(y) \frac{dy}{y}, x > 0 \qquad (2.1)$$

Where

$$k(x) = x^{(a-2)} . e^{-bx^{-1}} . H_n^{(-1)}(x,a-2,b)\left(b^n / n!\right)$$

and g is a prescribed function then f is given by

For

$$r=-1, p=b(b>0)$$

$$f(x) = (-1)^n x^{n-(a-2)} \int_0^\infty (b)^{-(n-2(a-2))} H_{3,2}^{0,2}\left[\frac{by}{x}\Big|_{(1,1)(1+n-(a-2),1)}^{(1+n,1)(1-(a-2)+2n,1)(2(n-(a-2)),1)}\right]\left[\left(\frac{d}{dy}\right)^n \left\{y^{(a-2)} g(y)\right\}\right]\frac{dy}{y}$$

3. SOLUTION

To Prove the Theorem we shall use of Mellin Transform and shell discuss case $r<0, p>0$

By the convolution theorem for Mellin Transform (2.1) reduces to

$$k^*(s) f^*(s) = g^*(s) \qquad (3.1)$$

Where $k^*(s), f^*(s), g^*(s)$ are respective Mellin Transform of $k(x), f(x), g(x)$ and by Sneddon

$$f^*(s) = \int_0^\infty f(x).x^{s-1}dx = M\left[f(x), s\right] \qquad (3.2)$$

When $r=-1$ and $p=b,$ Applying Mellin Transform of equation (3.1) and use the result of Erdelyi $k^*(s) = (-1)^n \frac{b^n}{n!} M\left[D^n\left(x^{(a-2)}e^{-bx^{-1}}\right); s\right]$ [12], we get

$$= \frac{\frac{b^n}{n!}\Gamma s\,(b)^{(s-n+a-2)}\,\Gamma\left(-(s-n+a-2)\right)}{\Gamma(s-n)}$$

$$= \frac{\Gamma s\,(b)^{(s+a-2)}\,\Gamma\left(-(s-n+a-2)\right)}{n!\,\Gamma(s-n)} \qquad (3.3)$$

Where

$$\mathrm{Re}(s) < n - \mathrm{Re}(a-2), \quad where \ \mathrm{Re}(a-2) \geq 0$$
$$\mathrm{Re}(s) < n, \quad where \ \mathrm{Re}(a-2) < 0$$

We write equation (3.1) in the form

$$f^*(s) = \frac{g^*(s)}{k^*(s)}$$

Replacing s by $s-n+a-2$

$$f^*(s-n+a-2) = (-1)^n L^*(s)\left[(-1)^n \frac{\Gamma s}{\Gamma(s-n)}.g^*(s-n+a-2)\right] \qquad (3.4)$$

Where

$$L^*(s) = \frac{\Gamma(s-n)}{\Gamma s.k^*(s-n+a-2)} \qquad (3.5)$$

Then from (3.3) and (3.5)

$$L^*(s) = \frac{(b)^{(s-n+2(a-2))}\,\Gamma(s-2n+a-2)\,\Gamma(s-n)}{\Gamma s\,\Gamma(s-n+a-2)\,\Gamma\left(-(s-2n+2a-4)\right)} \qquad (3.6)$$

$$\mathrm{Re}(s) < n - \mathrm{Re}(a-2), \quad where \ \mathrm{Re}(a-2) \geq 0$$
$$\mathrm{Re}(s) < n, \quad where \ \mathrm{Re}(a-2) < 0$$

By use of definition of H function. We get the inverse transform $L(x)$ of $L^*(s)$ as

$$L(x) = b^{-(n-2(a-2))} H_{3,2}^{0,2}\left[\frac{b}{x}\left|\begin{matrix}(1+n,1)(1-(a-2)+2n,1)(2(n-(a-2)),1)\\(1,1)(1+n-(a-2),1)\end{matrix}\right.\right] \qquad (3.7)$$

Where $H_{m,n}^{p,q}$ are Fox's H functions defined by [8].

And now taking Mellin Transform on both sides of (3.4), using convolution theorem and result of Mellin Transform

We get

$$M^{-1}\left[f^*(s-n+a-2)\right] = (-1)^n \int_0^\infty L(x)\left[M^{-1}\left\{(-1)^n \frac{\Gamma s}{\Gamma(s-n)} g^*(s-n+a-2)\right\}\right]$$

$$x^{a-2-n} f(x) = (-1)^n \int_0^\infty L(x)\left[\left(\frac{d}{dy}\right)^n \left\{y^{(a-2)} g(y)\right\}\right]\frac{dy}{y}$$

$$f(x) = (-1)^n x^{n-(a-2)} \int_0^\infty L(x)\left[\left(\frac{d}{dy}\right)^n \left\{y^{(a-2)} g(y)\right\}\right]\frac{dy}{y}$$

Hence using (3.7)

$$f(x) = (-1)^n x^{n-(a-2)} \int_0^\infty (b)^{-(n-2(a-2))} H_{3,2}^{0,2}\left[\frac{by}{x}\left|\begin{matrix}(1+n,1)(1-(a-2)+2n,1)(2(n-(a-2)),1)\\(1,1)(1+n-(a-2),1)\end{matrix}\right.\right]\left[\left(\frac{d}{dy}\right)^n \left\{y^{(a-2)} g(y)\right\}\right].\frac{dy}{y}$$

This proves the theorem.

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