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Some New Results on Strictly Singular Operators and Pre- Compact Operators

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Abstract – A persistent straight administrator T: E ⁇ F is viewed as absolutely particular if each limitless dimensional shut subspace of its area can't be invertible. In this notice, we address appropriate conditions and implications of the LB(E, F) = Ls(E, F) hypothesis, which implies that each continuous linear bounded operator described in E to F is strictly special. In the event that it maps an area of the source of E into a limited subset of F, a ceaseless direct administrator planning a neighborhood arched space (lcs) E into a lcs F is supposed to be limited If E or F is a uniform space, at that point any nonstop straight administrator is bound among E and F. A nonstop straight administrator T: E on the same page F is viewed as simply solitary on the off chance that it isn't invertible on some vast shut subspace of E. Kato[13] added simply particular administrators to the class of Banach spaces as per the bother rule of Fredholm administrators.

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INTRODUCTION

A compact administrator is only explicit, despite the fact that all in all the opposite isn't substantial. Van Dulst first concentrated carefully remarkable operators on Ptak (or B-complete) spaces and summed up Hilbert spaces as far as lcs. For the class of Br-complete spaces, Wrobel recognized carefully particular operators on lcs's. In the event that the pair (X, Y) has a place with the class of Banach spaces, Ls(X, Y) is a proficient administrator. This isn't the situation, as observed in [5], since it has a place with the overall class of lcs. However, each class of purely special bounded operators LBs (E, F) and compact operators Lc(E, F) is an ideal operator in lcs. By [22], if F has the property (y), for Fréchet spaces E and F, LBs (E, F) = Ls(E, F); If, as a quotient, E includes l1, then LBs(E, F) Ls(E, F).

Lemma 1. *Let E and F be lcs's where E is B_r-complete. Then, L_s(E, F) forms an operator ideal.*

Proof. Suppose that $T : E \to F$ and $S : E \to F$ are strictly singular operators. Then, for any $M \leq E$, by [28, Theorem 1-IV], find $N \leq M$ such that $T|_N$ is pre-compact. Then find $P \leq N$ such that $S|_P$ is pre-compact. The ideal property of pre-compact operators on lcs's yields the result.

We give the accompanying recommendation as a use of administrator ideal property of carefully particular operators on Br-complete lcs's. It is a speculation of [1, 2010 Mathematics Subject Classification. 46A03, 46A11, 46A32, 46A45.Problem 4.5.2], and in particular, it is also true when re-stated for bounded strictly singular operators acting on general lcs's.

Proposition 1. $E := \bigoplus^n E_i \ and\ F := \bigoplus^m F_j$ be lcs's where *E is B_r-complete. Then, $T : E \to F$ is strictly singular if f each of $T_{ij} : E_i \to F_j$ is strictly singular for each i = 1, 2, . . . , n and for each j = 1, 2, . . . , m.*

Proof. Assume that each T_{ji} is strictly singular. Let $\pi_i : E \to E_i$ be the canonical projection and define $\rho_j : F_j \to F$ by $\rho y_j = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus y_j \oplus 0 \oplus \cdots \oplus$ 0, for which yj is the j-th summand. Consider $E \xrightarrow{\pi_i} E_i \xrightarrow{T_{ji}} F_j \xrightarrow{\rho_j} F$ and write where Tji is the j-th summand. By Lemma 1 and rewriting $T = \sum_{i=1}^{n} \sum_{j=1}^{m} S_{ji},$

T is strictly singular. For the converse, let $T \in L_s(E, F)$, and suppose that the operator T_{ji} is not strictly singular for some i, j, and for $M \leq E$, $r \in I$ and $s \in J$, $N_{rs}(T|_M) := \sup\{q_s(Tx) : p_r(x) \leq 1, x \in M\}$. Then by [20, Theorem 2.1], for any $M \leq E_i$ and for some $s \in J$, $N_{rs}(T|_N) > \varepsilon$, for all $r \in I$. If we write M

$$\widehat{M} := \{0\} \oplus \{0\} \oplus \cdots \oplus M \oplus \{0\} \oplus \cdots \oplus \{0\}$$

Where *M places* in the *i*-th summand, M^ is a vector subspace of X. Nrs(τMˆ) > ε, for all r ∈ I. Yet, that negates the presumption T is carefully particular.

Each compact operator has a simple boundary. If we substitute compactness with absolute uniqueness, the results reveal that a purely singular operator T: E ⁇ F between lcs may not be unbounded, as such an inference contributes to an inconsistency with Bessaga, Pelczynski, and Rolewicz's well-known outcome. In Section 2 we survey the additional assumptions in which a bounded operator is immediately purely singular between two lcs.

STRICT SINGULARITY OF BOUNDED OPERATORS

In this segment, the extra speculations for a limited administrator's exacting peculiarity are included. Our beginning stage would be the Banach Spaces class. Utilizations of [7, Lemma 2] can bring about any of these discoveries getting significant for the lcs class. In a given case, it is important to accomplish a characterization (Theorem 3). For L(X, Y) = Ls(X, Y) in Banach spaces, the most generally perceived non-insignificant case is when X = lp and Y = lq with the end goal that 1 = p < q < ⁇ .This concept was important in the isomorphic classification of Cartesian power series spaces and in the issue of whether the sum of two supplemented subspaces is also supplemented[15] (if E subspaces are supplemented by X and Y, X+Y is supplemented by E if LB(X, Y) = Ls(X, Y)). If a weakly convergent sequence in X converges in norm, a Banach space X is said to have the Schur property (SP). On the off chance that a compelled arrangement is feebly Cauchy, X is viewed as practically reflexive. X is viewed as pitifully successively complete (wsc) if any feebly joined Cauchy arrangement in X is powerless. X is said to have the Dieudonné property (DP) if operators on X are feebly compact while changing frail Cauchy successions into pitifully joined groupings.

A couple of Banach spaces (X,Y) is considered completely unique if there is no Banach space Z isomorphic to a subspace of X and Y. A property P is viewed as innate on a Banach space X if any M ⁇ X cherishes it. X is expected to have P no place if the property P has no subspace. By Lw and Lv respectively we denote the groups of weakly compact and wholly continuous (or complete) operators. The Dunford-Pettis property (DPP) is said to have a Banach space X if Lw(X, Y) ⁇ Lv(X, Y), for each Banach space Y. X, if Lv(X, Y) ⁇ Lw(X, Y), is said to have the proportional Dunford-Pettis property (rDPP).

Lemma 2. Leave X and Y single Banach spaces.

1. Let X be practically reflexive.Then, for any wsc Banach space Y, L (X, Y) = Ls(X, Y).

2. Let X have DP, and let Y be wsc. Then, L(X, Y) = Lw(X, Y).

3. Let Y be almost reflexive. Then L(X, Y) = Lv(X,Y) implies L(X,Y) =Ls(X, Y).

4. Let X be a Banach space with SP. Then, for every M ≤ X, ℓ1 ⊙→ M.

Proof.

1. Because X is practically reflexive, at that point (Txn) has a feebly Cauchy grouping in Y if (xn) is a limited arrangement in X. But Y is wsc, that is, any weak sequence of Cauchy weakly converges in Y. T is weakly compact, along these lines.

2. Let (xn) x be Cauchy weakly, and let T be L(X, Y). Then (Txn) is Cauchy weakly in Y. Since Y is believed to be wsc, (Txn) weakly converges. X, though, has DP, so T x Lw(X, Y).

3. See [16, 1.7], Theorem.

4. Let X has the SP and concludes that M does not contain. Any bounded sequence (xn) in M then has a weak Cauchy subsequence because M is almost reflexive equivalently. M should, though, inherit SP. The poor Cauchy series of (xk) then converges into X. Therefore, M is dimensionally finite Disagreement.

Theorem 1. *Leave X and Y single Banach spaces. Every one of the accompanying suggests L(X, Y) = Ls(X, Y).*

1. X and Y are absolutely exceptional.

2. X is nowhere reflexive, Y is reflexive.

3. X is nowhere reflexive, Y is quasi-reflexive.

4. X is practically reflexive and no place reflexive, Y is wsc (see Example 1).

5. Y has hereditary P, X has nowhere P.

6. Y is almost reflexive, X is hereditarily-ℓ^1.

7. X has SP, Y is almost reflexive.

8. X is reflexive, Y has SP.

9. X has the hereditary DPP, Y is reflexive.

10. L(X, Y) = Lw(X, Y) and X has DPP.

11. L(X, Y) = Lv(X, Y) and X has rDPP.

12. X is a Grothendieck space with DPP, Y is separable.

13. X has both DP and DPP; Y is wsc (see Example 1).

Proof.

1. Assume T : X → Y is a non-strictly singular operator. So find M ≤ *X* on which $M \simeq T(M) \leq Y$. Since *X* and *Y* are totally incomparable, this is impossible.

2. See Theorem This result is generalized in part 5.

3. Suppose there exists a non-carefully particular administrator T ∈ L(X, Y). At that point, T is an isomorphism when limited to M ≤ X, so M is semi reflexive. In any case, by [12, Lemma 2] there exists a reflexive N ≤ M. This repudiates the presumption X is no place reflexive.

4. By section 1 of Lemma 2, L(X, Y) = Lw(X, Y). Presently let T : X → Y which has a limited backwards on M ≤ X. In the event that (xn) is a limited arrangement in M, at that point there exists (Txkn) a feebly joined aftereffect of (Txn) in Y . Henceforth (xkn) is feebly united in M, since T has a limited backwards on M.

In this manner, each limited grouping in M has a feebly merged aftereffect in M. So M is reflexive Contradiction.

5. For some M ≤ X assume there exists T: X → Y with the end goal that M ≃ T (M). Be that as it may, T (M) acquires P. Consequently M has P. This repudiates X has no place P. Presently let S: Y → X be with the end goal that for some N ≤ Y. Since X has no place P, S(N) abhors P . Logical inconsistency.

6. A specific instance of section 5.

7. Any administrator T with extends Y maps limited groupings into feebly Cauchy arrangements, since Y is practically reflexive. Then again, any such T de-fined on X maps pitifully Cauchy groupings into standard focalized successions by the SP. That infers L(X, Y) = Lv(X, Y). Subsequently by section 3 of Lemma 2 the outcome follows.

8. Let T ∈ L(X, Y) have a limited reverse on some M ≤ X, that is, (M). Since Y have SP, it likewise has the inherited DPP [6]. Thus does as well M. In any case, M is reflexive. By [14, Theorem 2.1], reflexive spaces have no place DPP. Logical inconsistency.

9. Let M ≤ X on which a self-assertive administrator T: X → Y has a limited opposite. At that point, So, M is reflexive. Henceforth M can't have DPP. Inconsistency.

10. Since X has DPP, L(X, Y) ∈ Lv(X, Y). At that point, by [16, Theorem 2.3], the outcome follows.

11. Since X has the rDPP and L(X, Y) = Lv(X, Y), T ∈ Lv ∩ Lw. By [16, Theorem 2.3], we are finished.

12. By [21, Theorem 4.9], any such administrator T: X → Y is pitifully compact. Since X has the DPP, T is totally ceaseless. By section 10, we arrive at the outcome.

13. By section 2 of Lemma 2, L(X, Y) = Lw(X, Y). X has the DPP, so L(X, Y) = Lv(X, Y). By [16, Theorem 2.3], the evidence is finished.

Example 1. Note that the non-reflexive space c_0 is almost reflexive. Suppose there exists a reflexive subspace *E* of c_0. Since c_0 fails SP, it is not isomorphic to any subspace of *E*. But this contradicts [17 Proposition 2.a.2]. The space *C*(*K*), where *K* is a compact Hausdorff space enjoys both DP and DPP .

Corollary 1. *Let X, Y, W, Z be Banach spaces. Then,*

1. If X', Y', Z' have SP and W is almost reflexive, every operator defined on $(X\hat{\otimes}_\pi Y)'$ *into* $W\hat{\otimes}_\pi Z$ *is strictly singular.*

2. If X and Y are reflexive spaces one of which having the estimate property, L(X, Y') = L (X, Y'), W and Z have SP, then every operator defined on $X\hat{\otimes}_\pi Y$ *into* $W\check{\otimes}_\varepsilon Z$ *is strictly singular.*

3. If X is almost reflexive and Y is almost reflexive and has DPP, then every operator defined on X⊗^ π Y into ℓ is strictly singular.

Proof.

1. By [16, Corollary 1.6], *L*(*W*, *Z*') = *L* (*W*, *Z*'). So by [9, Theorem 3] we deduce $W \otimes_\pi Z$ is almost reflexive. On the other hand, by [23, Theorem 3.3(b)] we reach that *L*(*X*, *Y*') has SP. But in [24] it is proved that $L(X,Y') \simeq (X\hat{\otimes}_\pi Y)'$. So $(X\hat{\otimes}_\pi Y)'$

SP. Therefore, Theorem 1 part 7 yields the result.

2. By [24, Theorem 4.21], $X\hat{\otimes}_\pi Y$ is reflexive. By [18], SP respects injective tensor products. So $W\otimes_\varepsilon Z$ has SP. Then, part 8 of Theorem 1 finishes the proof.

3. By [6], Y' has SP. Then by [16, Corollary 1.6], $L(X, Y') = L(X, Y')$. Hence, [9,

4. Theorem 3] yields that $\widetilde{X\hat{\otimes}_\pi Y}$ is almost reflexive. It is clear that every operator defined from an almost reflexive space into ℓ^1 is strictly singular.

5. Because X is almost reflexive, if (xn) in X is a small series, then (Txn) in Y has a poor Cauchy series. But Y is wsc, that is, any weak series of Cauchy converges in Y weakly. T is therefore weakly compact.

6. Let (xn) x be Cauchy weakly, and let T be L(X, Y). So (Txn) Cauchy is small in Y. Because Y is believed to be wsc, (Txn) is weakly converging. X, though, has DP, so T ⁇ Lw(X, Y).AlsoTheorem.

7. Let X have the SP and reason that M/X doesn't contain. Any limited arrangement (xn) in M at that point has a pitifully cauchy aftereffect since M is practically reflexive comparably. M acquires SP, however. The poor Cauchy arrangement of (xk) at that point unites with X. M is subsequently limited dimensional. The irregularity.

Let $\lambda_1(A) \in (d_2)$, and $\lambda_p(A) \in (d_1)$ as in [8]. Then, by [30], $L(\lambda_1(A), \lambda_p(A)) = LB(\lambda_1(A), \lambda_p(A))$. For $1 \le p < \infty$, we know that $\lambda_p(A)$ = proj lim $\ell^p(a_n)$. Since $\ell^p(a_n)$, $1 < p < \infty$ has no subspace isomorphic to ℓ^1, $L(\ell^1, \ell^p) = L_s(\ell^1, \ell^p)$. Then, by [7, Lemma 2], $L(\lambda_1(A), \lambda_p(A)) = L_s(\lambda_1(A), \lambda_p(A))$. Resting on the same argument, to obtain several sufficient conditions for $L(E, F) = L_s(E, F)$ is possible for the class of general lcs's.

Theorem 2. *Let E, F be lcs's. Each of the following implies* $LB(E, F) = L_s(E, F)$.

1. *E* ∈ s(X) *and F is locally Rosenthal.*

2. *E* ∈ s(V) *and F is a quasinormable Fréchet space.*

3. *E is infra-Schwartz and F* ∈ s(V)

4. *E* ∈ s($P^{\neg}$) *and F* ∈ s(P)

Proof.

1. Since F is locally Rosenthal, there exists a group of Banach spaces {Fm} every one of which doesn't contain an isomorphic duplicate of ℓ1 with the end goal that F = proj lim Fm. Because E ∈ s(X), there exists a family of Banach spaces $\{E_k\}$ such that every $M_k \le E_k$ contains a subspace isomorphic to ℓ^1. By part 6 of Theorem 1, any linear operator $T_{mk} : E_k \to F_m$ is strictly singular. Making use of [7, Lemma 2], we reach the result.

2. By [19, Theorem 6], F is locally Rosenthal. Since E ∈ s(V), by part 4 of Lemma 2, E ∈ s(X). Then, by part 1, we are done.

3. Since E is infra-Schwartz, any of its local Banach spaces E_k is reflexive.

The assumption on F completes the conditions in part 8 of Theorem 1. Combined with [7, Lemma 2], we are done.

4. Since E ∈ s($P^{\neg}$), one may rewrite E = proj lim E , where each E has no subspace having property P . Similarly, F = proj lim$m$ F_m where each F_m is hereditarily P . Hence, by part 5 of Theorem 1, $L(E_k, F_m) = L_s(E_k, F_m)$ for every k, m. Applying [7, Lemma 2], we obtain $LB(E, F) = L_s(E, F)$.

Theorem 3. 1. *Let (E, F, G) be a triple of Fréchet spaces fulfilling the accompanying*

1. *Every subspace of E contains a subspace Isomorphic to G.F has no subspace isomorphic to G.*

2. *Then,* $LB(E, F) = L_s(E, F)$. *Let F have continuous norm in addition. Then, is also necessary if F is a Fréchet-Montel space.*

Proof. The adequacy aspect is quite close to the proof of Part 4 of Theorem 2. Let E be a Fréchet space, for requirement, and let F be a (FM)-space admitting a continuous norm. Let each linear T operator be purely special. And it is bounded by [29, Proposition 1]. Let N Y, which is isomorphic to G, live now. And I: N is enclosed, compact, thus. All N is Dimensional Finite. Disagreement.

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