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Some General Relativistic Fluid Spheres

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Abstract – The present paper provides new solution of Einstein's field equations for the interior Metric of a fluid sphere with and without cosmological constant and by taking suitable conditions on g44. Various physical parameters have been found and discussed.

Key Words – Sphere, Pressure, Density, Metric Potential, Regular.

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1. INTRODUCTION:

Many research workers have shown their interest on finding solutions of Einstein's field equations in general relativity. Perfect fluid spheres with homogeneous density and isotropic pressure in general relativity were considered by Schwarzschild [16] and the solutions of relativistic field equation were obtained. Tolman [18] developed method for tracing Einstein's field equation applied to static fluid spheres in such a manner as to provide explicit solutions in terms of known analytic functions. A number of new solutions were thus obtained and the properties of three of them were examined in detail. These solutions were used by Open-heimer and Volkoff [14] in the study of massive neutron cores. Krori [5] obtained exact solutions for some dense massive spheres and pointed out their astrophysical implications. Krori [5] applied the solution V of Tolman in the case stellar structures with a variable-density core having finite density and pressure at the centre of the body. The characteristics of Tolman's solution VI with reference to constant density as well as variable density cores and their astrophysical implications have also been discussed. Mehra, Vaidya and Kushwaha [11] have obtained a general solution of the field equations for a complete sphere having a number of shells, one above the other, of different densities.

Durgapal and Gehlot [1] have obtained exact internal solutions for dense massive stars in which the control pressure and density are infinitely large. Durgapal and Gehlot [2], [3] have obtained exact solution for a massive sphere with two different density distributions. The density being minimum at the surface varies as the square of the distance from the centre. The distribution has a core of constant density and radius.

As a matter of fact solutions of Einstein's field equations in general relativity is much discussed problem. Solutions giving an isotropic and homogeneous distribution of matter in space have since long been known in differential geometry. Such solutions have special interest in general relativity as they afford suitable models of a universe which is assumed to consist of isotropic and homogeneous matter. Such a model was considered by Friendmann and Lemaitre in their solutions for the expanding universe. The well-known Schwarzschild interior solution [16] representing the field of a fluid sphere of constant density, was discovered long ago and still holds a prominent place in theory of relativity. Later on Whittaker [19] pointed out that the effective mass density governing gravitational attraction is not ρ but $\left(\rho + \frac{3p}{c^2}\right)$ whittaker [20] solved the Einstein's field equations for the interior metric of a fluid sphere assuming effective mass density to be a constant. However a more general case will involve a form of this quantity varying with redial co-ordinate. Krori et al. [6,7] found an internal solution for a spherically symmetric matter distribution with ρ c2 + 3p = f (r). Further on Paul and Guha Thakurta [15] studied the same problem taking cosmological constant into account.

A method for treating Einstein's field equations applied to static sphere of fluid to provide solutions in terms of known analytic functions was developed by Tolman [18]. Leibovitz [9], [10] has extensively discussed the static and non-static solutions of Einstein's field equations for the spherically symmetric distributions. The significance of the Weyl conformed curvature tensor in relation to distribution of spherical symmetry, has been investigated by Narlikar and Singh [13]

Hargreaves [4] has discussed the stability of a static spherically symmetric fluid spheres, consisting of a core of ideal gas and radiation, in which the ratio of the gas pressure to the total pressure is a small constant, and an envelope consisting of an adiabatic gas. Yadav and Saini [21] have obtained an exact, static spherically symmetric solution of Einstein's field equation for the perfect fluid with p = p while Leon [8] has presented two new exact analytical solutions to Einstein's field equations representing static fluid spheres with an isotropic pressure. Some other workers in this field are steward [17] and Yadav et. at [22].

Here in this paper we have presented a new solution of the Einstein's field equation for the interior metric of a fluid sphere taking cosmological constant in the solution. We have assumed that the metric coefficient b = αr2 where -b = g44' so that the effective mass density varies with radial co-ordinate.

We have also considered the case when cosmological term λ is zero and b = Ar2 + B.

2. THE FIELD EQUATIONS AND THEIR SOLUTIONS

We assume a metric of the form

(2.1) $$ds^2 = a(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - b(r)c^2dt^2$$

The equations to be satisfied are them (Moller [12])

(2.2) $$\frac{dp}{dr} + (\rho c^2 + p)\frac{b'}{2b} = 0$$

(2.3) $$\frac{b'}{abr} - \frac{1}{r^2}\left(1 - \frac{1}{a}\right) + \lambda = kp.$$

(2.4) $$\frac{a'}{a^2r} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) - \lambda = k\rho c^2$$

Where λ is the cosmological constant and prime denotes differentiation with respect to r. Equations (2.2) - (2.4) are three equations in four unknowns a, b, p and p. Thus the system is inderminate. For complete determinacy of this system we should have one more relation. For this, we assume following cases.

Case – I

(2.5) $$b = \alpha r2$$

Where α is an arbitrary constant

Now from equations (2.5) and (2.3), We have

(2.6) $$\frac{3}{ar^2} - \frac{1}{r^2} + \lambda = kp$$

Adding equations (2.6) and (2.4) we get

(2.7) $$\frac{a'}{a^2r} + \frac{2}{ar^2} = k(\rho c^2 + p)$$

Equations (2.5) and (2.7) together give

(2.8) $$k(\rho c^2 + p)\frac{b'}{2b} = \frac{a'}{a^2r^2} + \frac{2}{ar^3}$$

Equations (2.6) on differentiation yields

(2.9) $$-\frac{3a'}{a^2r^2} - \frac{6}{ar^3} + \frac{2}{c^3} = dp'$$

Which by use of (2.8) and (2.2) reduces to

(2.10) $$-\left(\frac{a'}{a^2r^2} + \frac{2}{ar^3}\right) + \frac{1}{r^3} = 0$$

Which on integration further reduce to

(2.11) $$\frac{1}{a} = \eta r^2 + \frac{1}{2}$$

From equations (2.6) and (2.11) pressure is given by

(2.12) $$kp = 3\eta + \frac{1}{2r^2} + \lambda$$

and from equations (2.4) and (2.11)

(2.13) $$k\rho c^2 = -3\eta + \frac{1}{2r^2} - \lambda$$

and from (2.12) and (2.13)

(2.14) $$k(\rho c^2 + 3p) = 6\eta + \frac{2}{r^2} + 2\lambda$$

Clearly $(\rho c^2 + 3p)$ is a variable quantity depending upon the radial co-ordinate r. It can be seen from equations (2.12) that p increases as r decreases. It becomes zero for r1 given by

(2.15) $$\frac{1}{r_1^2} = -2(3\eta + \lambda)$$

Clearly $(3\eta+\lambda)$ is a negative quantity.

At the boundary the interior solution passes, over to the exterior solution so that we have (Moller [12]).

$$b(r_1)=\{a(r_1)\}^{-1}=1-\frac{2m}{r_1}-\frac{1}{3}\lambda r_1^2 \tag{2.16}$$

From equations (2.5), (2.11) and (2.16) we have the following values for m

$$m=\frac{r_1}{2}\left(1-\alpha r_1^2-\frac{1}{3}\lambda r_1^2\right) \tag{2.17}$$

$$m=\frac{r_1}{2}\left(\frac{1}{2}-\eta r_1^2-\frac{1}{3}\lambda r_1^2\right) \tag{2.18}$$

From (2.17) and (2.18) we have

$$\eta=\alpha-\frac{1}{2r_1^2} \tag{2.19}$$

Thus the value of η is fixed by the values of α and r1.

Case – II: Here we assume cosmological term λ = 0 and

$$b=Ar^2+B \tag{2.20}$$

where A and B are constants. Adding equations (2.3) and (2.4) we get

$$\rho c^2+p=\frac{1}{8\pi ar}\left(\frac{a'}{a}+\frac{b'}{b}\right) \tag{2.21}$$

The equation (2.21) can be rewritten as

$$\rho C^2+3p=\frac{1}{8\pi ar}\left(\frac{a'}{a}+\frac{b'}{b}+\frac{1}{r}-\frac{b''}{b'}\right)+2p \tag{2.22}$$

Since

$$\frac{1}{r}-\frac{b''}{b'}=0$$

From equations (2.20) and (2.2) using equation (2.22) we get

$$\frac{dp}{dr}+\left[\frac{1}{8\pi ar}\left(\frac{a'}{a}+\frac{b'}{b}+\frac{1}{r}-\frac{b''}{b'}\right)\right]\frac{b'}{2b}=0$$

which on integration leads to

$$p=\frac{1}{16\pi}\left(\frac{b'}{abr}\right)+c \tag{2.23}$$

where C is the constant of integration. From equation (2.3) using equation (2.20) and (2.23) we get

$$a=\frac{Ar^2+B}{(1+8\pi cr^2)\left(\frac{1}{2}Ar^2+B\right)} \tag{2.24}$$

Hence from equation (2.23) using the values of a and b from (2.24) and (2.20) respectively we get

$$p=\frac{1}{16\pi}\frac{1+8\pi cr^2}{r^2+(B/A)} \tag{2.25}$$

And from equation (2.4) using equation (2.24) we get

$$\rho c^2=\frac{1}{8\pi}\left(\frac{-12A^2Cr^4+\left(\frac{1}{2}A^2-28\pi ABC\right)r^2+\left(\frac{3}{2}AB-24\pi A^2C\right)}{(Ar^2+B)^2}\right) \tag{2.26}$$

From equation (2.25) and (2.26) we get ρc^2+2p as a variable quantity.

$$\rho C^2+3p=\frac{1}{16\pi}\left(\frac{48\pi A^2Cr^4+(4A^2+64\pi ABC)r^2+6AB}{(Ar^2+B)^2}\right) \tag{2.27}$$

Since at the boundary p = 0 we get from the equation (2.25)

$$r_1=\left(\frac{-(16\pi BC+A)}{24\pi AC}\right) \tag{2.28}$$

Where r1 is the boundary. To make r1 real c should be negative and $A>16\pi|c|$. with the above conditions ρc^2, p and ρc^2+ 3p are all positive and their values at the centre of the sphere are

$$P_0=\frac{1}{16\pi}\frac{A}{B}+C \tag{2.29}$$

$$P_0c^2=\frac{3}{16\pi}\frac{A}{B}-3C \tag{2.30}$$

$$\rho_0c^2=3p_0=\frac{3}{8\pi}\frac{A}{B} \tag{2.31}$$

Hence from above it shows that ρ_0c^2, p_0 and $\rho_0c^2+3p_0$ are all positive. Therefore from equation (2.29) and (2.30), since c is negative

(2.32) $$\rho_0 c^2 > 3 \cdot p_0$$

Now for the exterior solution we know (Moller [12]).

(2.33) $$b(r) = \frac{1}{a(r)} = 1 - \frac{2m}{r}$$

As a(r) and b(r) must be continuous and ab = 1 for the exterior solution, we have using equation (2.20) and (2.24)

(2.34) $$1 = \frac{Ar_1^2 + B}{1 + 8\pi c r_1^2}$$

From which we get

(2.35) $$c = -\left(\frac{\left[(2A - AB)^2 + BA^2B \right]^{1/2} - (2A - AB)}{32\pi B} \right)$$

This value of C makes r1 real and the condition $A > 16\pi |C| B$ is satisfied, provided $1 > B > 0$. Hence from equation (2.33) and (2.34)

(2.36) $$\frac{2m}{r_1} = \frac{1}{2} A r_1^2 - 8\pi c r_1^2$$

It shows that m is positive since C is negative. It can be also shown that equation (2.36) can be expressed in terms of pressure and density as follows:

(2.37) $$\frac{2m}{r_1} = 8\pi r_1^2 \left[\rho_0 p_0 - \left\{ \frac{2}{3} p_0 - \frac{1}{6} (\rho_0 c^2 + p_0) \right\} (b_0 + 1) \right]$$

Where b0 is the value of b at r = 0

3. REMARK

Yet in case I our solution is not regular at the centre (r=0), the density and pressure vary with radial co-ordinate in a much simpler manner then in the solutions due to whittaker [20], Krori et at. [6, 7] and Paul and Guha Thakurta [15].

In case II our solution is regular at the centre r = 0 and also pressure, density both are positive.

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