A Study of Common Fixed Point Theorem in Cone Metric Spaces

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Abstract – In proving solutions of equation, the Banach theorem has several limitations. In particular, mapping should be continuous, and therefore it does not apply to nonlinear issues where the mapping is interruptive. In 1968, Kannan published a certain point theorem that spreads the Banach theorem. This breakthrough took place. He showed it is not necessary to continue the mapping which is contractive. Following the result of Kannan, there have been so many generalizations of fixed point theorems in Banach. The theorems based on Brouwer and Schauder are the most important results in the fixed point theory. These theorems are used to demonstrate the existence of differential, integral and differential solutions. Krasnoselskii, which obtained the set points of sum of two operators, further generalizes those theorems.Common fixed point theorems for general contractions always need a condition of commutativity and the continuity of one of the mappings. One or more of these conditions are weakened by research. In the study of problems of the common fixed points of non-commuting mapping, the notion of compatibility plays an important role.

Key Words – Common Fixed Point Theorem, Cone Metric Spaces, solutions of equation, nonlinear issues, non-commuting mapping

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INTRODUCTION

This study has been divided in three sections. In the first section we have proved some results on fixed point theorem in complete cone metric space for commutative mapping which is an extension.The idea for the above has been taken from the work done. Second section consists of two theorems. First theorem has been proved by using orbitally continuous mapping and finite number of function which is studied. Second theorem has been proved by using four mappings and finite number of function. In the third section we have proved two theorems in cone rectangular metric space. First theorem is an extended work of Azam Akbar, Arshad and Beg. In the second theorem we have established a result by using rational type contractive condition which is studied by Jaleli Mohamed and SametBessem.

1. COMMON FIXED POINT THEOREM AND THEIR PROPERTIES IN CONE METRIC SPACE

Definition 1.1. If E be a real Banach space and a nonempty set X. Suppose that the mapping satisfies

$$0 \le d(x,y) \text{ for all } x, y \in X \tag{1.1a}$$

$$d(x,y) = 0 \text{ if and only if } x = y \tag{1.1b}$$

$$d(x,y) = d(y,x) \text{ for all } x, y \in X \tag{1.1c}$$

$$d(x,y) \le d(x,z) + d(y,z) \text{ for all } x, y, z \in X. \tag{1.1d}$$

Then distance d is called a cone metric on *X* and set *X* with cone metric d is called cone metric space (X, d) .

A sequence $\{x_n\}$ in a cone metric space X converges to x if and only if

$$d(x_n, x) \to 0 \text{ as } n \to \infty \tag{1.2}$$

Let a sequence $\{x_n\}$ in a cone metric space X converges to x. If $\{x_n\}$ converges to y then $x = y$. That is limit of $\{x_n\}$ is unique.

A sequence $\{x_n\}$ is cone metric space X is said to be a Cauchy sequence, if for any $c \in E$ with $0 << c$ there is N such that

$$d(x_n, x_m) << c \text{ for all } n, m > N. \quad (1.3)$$

A sequence $\{x_n\}$ in cone metric space X converges to x, then $\{x_n\}$ is called a Cauchy sequence.

Let (X,d) be a complete cone metric space and f and g be two self mappings on X. If

$$w = fx = gx \text{ for some } x \text{ in } X. \quad (1.4)$$

Then x is called a coincidence point f and g and w is called a point of coincidence of f and g.

Two self-mapping *f* and *g* of a set *X* are said to be weakly compatible if they commute at their coincidence point, that is, if

$$fu = gu \text{ for some } u \in X \text{ then} \quad (1.5a)$$

$$fgu = gfu \quad (1.5b)$$

Let *f* and *g* be weakly compatible self-mapping of a set *X*. If *f* and *g* have a unique point of coincidence, that is $w = fx = gx$, then w is the unique common fixed point of *f* and *g* .

2. COMMON FIXED POINT THEOREM IN CONE RECTANGULAR METRIC SPACE

Definition 2.1. let X be a nonempty set and E be a real Banach space. Suppose the mapping $d : X \times X \to E$ satisfies

$$0 \le d(x,y), \forall x, y \in X. \quad (2.1a)$$

$$d(x,y) = 0 \text{ if and only if } x = y. \quad (2.1b)$$

$$d(x,y) = d(y,x) \ \forall x, y \in X. \quad (2.1c)$$

$$d(x,y) \le d(x,w) + d(w,z) + d(z,y), \forall x, y \in X \quad (2.1d)$$

and for all distinct point $w, z \in X - \{x, y\}$ (rectangular property).

Then d is called a cone rectangular metric on X and (X,d) is called cone rectangular metric space.

Lemma 2.1. A sequence $\{x_n\}$ in cone rectangular metric space X is said to be convergent if for every $c \in E$ with $0 << c$ there is $n_0 \in N$ such that

$$d(x_n, x) << c \text{ for all } n > n_0. \quad (2.2)$$

Lemma 2.2. A sequence $\{x_n\}$ in cone rectangular metric space X is said to be Cauchy if for every $c \in E$ with $0 << c$ there is $n_0 \in N$ such that

$$d(x_n, x_m) << c \text{ for all } n, m > n_0. \quad (2.3)$$

If every Cauchy sequence in cone rectangular metric space X is convergent then X is said to be complete cone rectangular metric space.

For $c \in E$ with $0 << c$ there is $n_0 \in N$ such that for all $n, m > n_0$ $d(x_n, x_m) << c$ then $\{x_n\}$ is called Cauchy sequence.

A cone rectangular metric space is said to be complete cone rectangular metric space if every Cauchy sequence in X is convergent.

Theorem 2.1. Let (X,d) is a complete cone rectangular metric space and P is a normal cone with normal constant K. Let f is self mapping from X into itself satisfying

$$d(fx, fy) \le \alpha \left\{ \frac{d(x,y) + d(x,fx) + d(fx,y)}{2} + d(y,fy) \right\} \quad (2.4)$$

$$\forall x, y \in X, \alpha \in [0,1) \text{ and } 0 < \frac{\alpha}{1-\alpha} < 1$$

then f has an unique fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Let us take a sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n = f_{x_0}^{n+1}, n \in N \cup \{0\}$$

Now substituting $x = x_0$ and $y = x_1$ in (2.4) we obtain

$$d(fx_0, fx_1) = d(x_1, x_2)$$

$$\le \alpha \left\{ \frac{d(x_0,x_1) + d(x_0,fx_0) + d(fx_0,x_1)}{2} + d(x_1, fx_1) \right\}$$

$$\le \alpha \left\{ \frac{d(x_0,x_1) + d(x_0,x_1) + d(x_1,x_1)}{2} + d(x_1, x_2) \right\}$$

$$d(x_1, x_2) \le \alpha \{ d(x_0, x_1) + d(x_1, x_2) \}$$

$$d(x_1, x_2) \le \frac{\alpha}{1-\alpha} d(x_0, x_1)$$

$$d(x_1, x_2) \le h d(x_0, x_1) \quad (2.5)$$

where $0 < h = \dfrac{\alpha}{1-\alpha} < 1$

Again for $x = x_1, y = x_2$ we have

$$d(x_2, x_3) = d(fx_1, fx_2)$$

3. FIXED POINT THEOREM IN COMPLETE CONE METRIC SPACES

Definition 1.1: E is a nonempty and real Banach space. Let P be a subset of E . The subset P be called a cone iff,

$$P \text{ is closed and nonempty and } P \neq \{0\} \quad (1.1a)$$

If u and $v \in P$ and $a, b \in R$ then $au + bv \in P$

where $$a, b \ge 0. \quad (1.1b)$$

$$P \cap (-P) = \{0\} \quad (1.1c)$$

By given a cone $P \subseteq E$, we define a partial ordering '≤' with respect to P.

$x \le y$ if and only if $y - x \in P$, $x < y$ indicate that $x \le y$ and $x \ne y$ while $x << y$ stands for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called a normal cone if there exists a number $k > 0$ such that $0 \le x \le y$ implies that $\| x \| \le k \| y \| \forall x, y \in E$. The least positive number satisfies the above condition is called the normal constant of P.

Definition 1.2. Let X be a nonempty set. Suppose that the mapping $X \times X \to E$ satisfying

$$d(x,y) \ge 0 \ \ \forall x, y \in X \tag{1.2a}$$

$$d(x,y) = 0 \text{ iff } x = y \tag{1.2b}$$

$$d(x,y) = d(y,x), \forall x, y \in X \tag{1.2c}$$

$$d(x,y) \le d(x,z) + d(z,x), \forall x, y \in X \tag{1.2d}$$

Then d is called a cone metric on X and (X,d) is called cone metric space.

Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 << c$ there is N such that for all $n > N$.

$d(x_n, x) << c$ then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x and the limit of $\{x_n\}$.

Let (X,d) be a cone metric space. A sequence $\{x_n\}$ in X is said to be Cauchy sequence if there is a c in E and $c >> 0$, there is N such that for all $n, m > N, d(x_n, x_m) << c$.

Theorem (1.1). Let (X,d) is a complete cone metric space and P be a normal cone with normal constant K. If f and g are self mapping from X into itself satisfying the condition.

$$d(fx, gy) \le \alpha \left[\frac{d(x,y) + d(x, fx)}{2} + d(y, gy) \right] \tag{1.3}$$

$\forall x, y \in X, \alpha \in (0,1)$ and $0 < \frac{\alpha}{1-\alpha} < 1$. Then f and g has an unique fixed point in X

Proof. Let x_0 is an arbitrary point in X. Let $\{x_{2n}\}$ is a sequence in X. So that we define the mapping f and g in X such that,

$$x_{2n-1} = f x_{2n-2}, n \in N \text{ and} \tag{1.4}$$

$$x_{2n} = g x_{2n-1}, n \in N \tag{1.5}$$

First we show that $\{x_{2n}\}$ is a Cauchy sequence in X. Now substituting $x = x_{2n-2}, y = x_{2n-1}$ in equation (1.3) we obtain,

$$d(x_{2n-1}, x_{2n}) = d(fx_{2n-2}, gx_{2n-1}) \tag{1.6}$$

$$\le \alpha \left[\frac{d(x_{2n-2}, x_{2n-1}) + d(fx_{2n-2}, x_{2n-2})}{2} + d(x_{2n-1}, gx_{2n-1}) \right]$$

$$= \alpha \left[\frac{d(x_{2n-2}, x_{2n-1}) + d(fx_{2n-1}, x_{2n-2})}{2} + d(x_{2n-1}, x_{2n}) \right]$$

$$\Rightarrow d(x_{2n}, x_{2n-1}) \le \frac{\alpha}{1-\alpha} d(x_{2n-3}, x_{2n-2}) \tag{1.7}$$

Again substituting $x = x_{2n-3}, y = x_{2n-2}$ in (1.3) we get,

$$d(x_{2n-2}, x_{2n-1}) = d(fx_{2n-3}, gx_{2n-2})$$

$$\le \alpha \left[\frac{d(x_{2n-3}, x_{2n-2}) + d(x_{2n-3}, fx_{2n-3})}{2} + d(x_{2n-2}, gx_{2n-2}) \right]$$

$$\Rightarrow d(x_{2n-3}, x_{2n-2}) \le \frac{\alpha}{1-\alpha} d(x_{2n-2}, x_{2n-3}) \tag{1.8}$$

From (1.3) and (1.8) we have,

$$d(x_{2n}, x_{2n-1}) \le \left[\frac{\alpha}{1-\alpha} \right]^2 d(x_{2n-2}, x_{2n-3}) \tag{1.9}$$

Continuing this process we get in general get in general,

$$d(x_{2n}, x_{2n-1}) \le \left[\frac{\alpha}{1-\alpha} \right]^n d(x_0, x_1) \tag{1.10}$$

$$\Rightarrow d(x_{2n}, x_{2n-1}) \le h^n d(x_0, x_1)$$

where, $h = \frac{\alpha}{1-\alpha} < 1$

Now for $n > p$ we have,

$$d(x_{2n+1}, x_{2n+p}) \le d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+p})$$

$$\le d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, x_{2n+p})$$

$$\le h^n d(x_0, x_1) + h^{n+1} d(x_0, x_1) + \cdots\cdots + h^{n+p}(x_0, x_1)$$

$$\Rightarrow d(x_{2n}, x_{2n+p}) \le h^n \{1 + h + h^2 + \cdots\cdots + h^p\} d(x_0, x_1)$$

$$\le \frac{h^n}{1-h} d(x_0, x_1) \tag{1.11}$$

Now applying the condition for the normality of cone we get from (1.11)

COMPARISON FUNCTION AND FIXED POINT THEOREM IN COMPLETE CONE METRIC SPACES

Definition. Let ϕ is a mapping from $R^+ \to R^+$, which satisfies the following conditions:

$$\phi \text{ is increasing} \tag{3.1a}$$

$$\phi(t) < t, \forall t > 0 \tag{3.1b}$$

$$\lim_{n \to \infty} \phi^n(t) = 0, \text{ where } \phi^n(t) \tag{3.1c}$$

denotes the composition of $\phi(t)$ with itself n-times.

Theorem (3.1). Let (X,d) be a complete cone metric spaces and P be a normal cone with normality of P is K. Let f is a mapping from X into itself. Also let φ is comparison function from P into P such that,

$$\varphi\{d(fx,fy)\} \leq \alpha[\varphi\{d(x,y)\} + \varphi\{d(fx,x)\} + \varphi\{d(y,fy)\}] \qquad (3.2)$$

$\forall x,y \in X$ and $\frac{2\alpha}{1-\alpha} \in (0,1)$ then f has a common fixed point.

Proof. Let x_0 be an arbitrary point of X. Let us take a sequence $\{x_n\}$ in X such that,

$$x_n = fx_{n-1} = fx_0, \forall n \in N.$$

Now,

$$\varphi\{d(x_n,x_{n+1})\} \leq \alpha\varphi\{d(fx_{n-1},fx_n)\}$$

$$\leq \alpha[\varphi\{d(x_{n-1},x_n)\} + \varphi\{d(fx_{n-1},x_{n-1})\} + \varphi\{d(x_n,fx_n)\}]$$

$$\leq \alpha\varphi\{d(x_{n-1},x_n)\} + \varphi\{d(x_n,x_{n-1})\} + \varphi\{d(x_n,x_{n+1})\}]$$

$$\Rightarrow \varphi\{d(x_n,x_{n+1})\} \leq \frac{2\alpha}{1-\alpha}[\varphi\{d(x_{n-1},x_n)\}$$

$$\Rightarrow \varphi\{d(x_n,x_{n+1})\} \leq h[\varphi\{d(x_{n-1},x_n)\} \qquad (3.3)$$

where $h = \frac{2\alpha}{1-\alpha} \in (0,1)$.

Again substituting $x = x_n$ and $y = x_{n+1}$ we get from the inequality condition (3.2) we obtain,

$$\varphi\{d(x_{n+1},x_{n+2})\} \leq \alpha\varphi\{d(fx_n,fx_{n+1})\}$$

$$\leq \alpha[\varphi\{d(x_n,x_{n+1})\} + \varphi\{d(fx_n,x_n)\} + \varphi\{d(x_{n+1},fx_{n+1})\}]$$

$$\leq \alpha[\varphi\{d(x_n,x_{n+1})\} + \varphi\{d(x_{n+1},x_n)\} + \varphi\{d(x_{n+1},x_{n+2})\}]$$

$$\Rightarrow \varphi\{d(x_{n+1},x_{n+2})\} \leq \frac{2\alpha}{1-\alpha}[\varphi\{d(x_n,x_{n+1})\} \leq h[\varphi\{d(x_n,x_{n+1})\}]$$

$$\varphi\{d(x_{n+1},x_{n+2})\} \leq h^2[\varphi^2\{d(x_{n-1},x_n)\}] \qquad (3.4)$$

Applying the above procedure we get in general $n > m$,

$$\varphi\{d(x_{n+m},x_{n+m+1})\} \leq h^n[\varphi^n\{d(x_0,x_1)\}] \qquad (3.5)$$

Now using the normality of cone, implies that,

$$\|\varphi\{d(x_{n+m},x_{n+m+1})\}\| \leq K\|h^n[\varphi^n\{d(x_0,x_1)\}]\|$$

Now letting $n,m \to \infty$ we have, $\|h^n[\varphi^n\{d(x_0,x_1)\}]\| \to 0$.

$$\Rightarrow \|\varphi\{d(x_{n+m},x_{n+m+1})\}\| \to 0$$

i.e., $\{x_{n+m}\}$ is Cauchy sequence in X i.e. $\{x_n\}$ is a Cauchy sequence.

CONCLUSION

The fixed point theory for multivalue operators in metric spaces was used in a number of works published in specialist literature, starting with a multivalued-view version of the Banach-Caccioppoli contraction principle demonstrated by S. B. Nadler Jr. of 1969. The development of this theory led to the creation of different applications in many fields, including: optimization theory, integral and differential equations and inclusions, fractal theory, econometrics, etc. One with remarkable applications is also the Avramescu-Markin-Nadler theorem, among fixed-point theorems with multi-value applications. Applications identified and demonstrated as the theorem for fixed points were: fixed point theorems for Kasahara, fixed point function theorems for Darboux, etc. Fixed point theorems in fudge metric spaces, which are included in a large number of works published in the specialty literature, are other significant results with many applications in fixed point theory. Fixed-point theories are important not only to the theory of differential equations, integrated equations, differential integrations and inclusions, but also in economic and managerial sciences, in the fields of computer science and in other areas.A new direction of research includes a new operator type, namely the α-to contractive operator. Published works include, for example, w-distance conditions. The problem here is that concrete examples for the w-distance are difficult to find since it is a more abstract concept. Asl, Rezapour, and Shahzad (2012) created the concept of α-t-al-purpose multinationals and showed how fixed point results could be achieved for this new type. Guran and Bota (2015) studied the existence of a fixed point of the α-to–constractive type of operator on KST space, its uniqueness and generalized Ulam-Hyers stability. A new problem is the establishment of the conditions in which there would be and would be uniquely fixed α-técontractive type operators, which is the problem of this type of vector contractions. The theory of fixed points in the last decades has been widely applied. It applies highly to optimisation theory, gaming theories, conflicts, but also to mathematical quality modeling and its management. These applications are useful and interesting.

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