Viscous Heating Effects of a Bingham Plastic Flowing Between the Parallel Plates

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Abstract – In this paper, my study is about the Bingham plastic fluid model. The fluid flow between two given parallel plates, both plates are at rest has been focussed in this work. Here, It is describing the theoretical analysis of the Bingham plastic fluid model between the plates. The equation of motion, energy equation and also continuity equation are solved simultaneously at the rate of the plates are of same temperature. Incompressible Bingham plastic fluid through parallel plates and semi analytical solution has been developed and the flow behavior and temperature profiles (temperature effects) has been demonstrated and represented with respect to different parameter, as well as the dissipation terms are examined.

Keywords: Temperature Dependent Viscosity, Viscosity Profile, Bingham Plastic, Temperature Effects, Flow rate, Eigen values, Eigen Function, Expansion Coefficients.

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1. INTRODUCTION

The Bingham plastic fluids are distinguished by the non-Newtonian fluids in that they require a finite stress to initiate flow. Bingham plastic (or simply Bingham) fluid is represented by a straight line in rheograms, but the lines does not pass through the origin. It takes a certain minimum shear stress, called the yield stress R_y to cause a Bingham fluid to behave like a fluid. For r_{yx} less than R_y a Bingham plastic fluid behaves like a solid rather than a fluid [1, 2]. When R_{yx} becomes greater than R_y, behaves like a Newtonian fluid [3]. Examples of Bingham plastic fluids include Water suspensions of clay, Fly ash, Sewage sludge, Paint. The development of modern engineering, lubrication technology, biophysics, biomechanics, solid mechanics and other branches of science and technology, which deal with high polymers, suspensions, pastes, oils, lubricants and physiological fluids, have made the study of non-Newtonian fluids important. The frequent occurrence of these fluids in industries and in day-to-day life have provided a great impetus to the detailed study of their flow behavior [4]

Sarpkaya [5] discussed the analytical solution of the equation and used the non-Newtonian fluids flow between two parallel plates. Velocity profiles with respect to different parameter has been discussed by Rashidi [6].

Szeri & Rajgopal [7] examined the fluid flow between two parallel plates. Siddiqui [8] discussed the fluid flow model between two parallel plates and explained the heat and mass transfer.

It is well known that the viscosity of liquids in viscometric experiments shows dependence on the rates of shear. An explanation of this is that high rates of shear result in high energy dissipation and hence there will be a temperature rise. Therefore in the interpretation of viscometric experiments, it is important to know how severe viscous heating effects are.

Thus Brinkman [9] has discussed this problem for Newtonian fluids and has calculated the temperature distribution by taking into account, the energy dissipation due to viscous heating. Later Bird [10], Sehenk and Van laan [11] have studied this problem for certain non-Newtonian fluids.

In this paper we have discussed the viscous heating effects of a Bingham plastic flowing between two parallel plates given by the equation y = –b and y = + b when these are thermally insulated.

The rheological equation of Bingham plastic as defined by Oldroyd [12, 13] is

$$e_{ij} = 0 \text{ if } \frac{p_{ik}p_{ik}}{2} \le T_y^2 \qquad (1.1)$$

$$p_{ij} = 2\left[\mu + \frac{T_y}{\sqrt{2e_{ij}e_{ik}}}\right]e_{ij} \text{ if } \frac{p_{ik}p_{ik}}{2} > T_y^2 \qquad (1.2)$$

2. ASSUMPTIONS AND FORMULATION OF THE PROBLEM

In the analysis of this problem, the following assumptions are made.

(a) Flow is laminar and fully developed from the instant of entering the tube.

(b) Free convection effects are ignored.

(c) The temperature rise is small such that the material properties are assumed to remain constant.

(d) Elastic energy of deformation is ignored.

The velocity profile of such a material flowing between two parallel plates under a constant axial pressure gradient with axial symmetry is given by

$$\frac{u}{u_{max}} = 1, \quad 0 \le \frac{y}{b} < c, \qquad (2.1)$$

$$1-\left(\frac{y}{b}\right)^2 - 2c\left(1-\frac{y}{b}\right)$$

$$\frac{u}{u_{max}} = \frac{1-\left(\frac{y}{b}\right)^2 - 2c\left(1-\frac{y}{b}\right)}{(1-c)^2}, c \le \frac{y}{b} < 4 \qquad (2.2)$$

Where, u is the axial velocity, u_{max} is the maximum axial velocity, b is the half width between plates and c is the ratio of yield stress to wall stress.

By assumptions (a) and (b), the energy equation may be written as

$$\rho c_v - u\frac{\delta T}{\delta x} = \left(\frac{\delta q_x}{\delta x} + \frac{\delta q_y}{\delta y}\right) + pxy\frac{du}{dy} \qquad (2.3)$$

Where, C_v is the specific heat at constant volume, T is temperature and q_x, q_y are axial and transverse components of heat flux vector.

Using the Fourier's law of heat conduction, viz

$$\vec{q} = \frac{-k\bar{\nabla}T}{\bar{\nabla}} \qquad (2.4)$$

where k is the thermal conductivity of the material.

we get

$$\rho C_v u\frac{\delta T}{\delta x} = k\left(\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2}\right) + p_{xy}\frac{du}{dy} \qquad (2.5)$$

The term on the left hand side represents convection of heat into an element of the fluid, the first two terms on the right hand side represent the conduction of heat into the element and the last term represents generation of heat by viscous dissipation. Since in most cases conduction in the axial direction is negligible compared to the flow of heat in the same direction due to convection, we neglect the term $k\frac{\delta^2 T}{\delta x^2}$ in the equation (2.5). Thus the energy equation reduces to

$$\rho C_v u\frac{\delta T}{\delta x} = k\frac{\delta^2 T}{\delta y^2} + p_{xy}\frac{dx}{dy} \qquad (2.6)$$

Let T_0 be the uniform temperature at which the fluid enters the pipe. Also the wall is thermally insulated so that the heat flux is zero at the walls. Therefore the boundary conditions are

$$T(0,y) = T_0 \qquad (2.7)$$

$$\frac{\delta T}{\delta y}(x,b) = 0 \qquad (2.8)$$

$$\frac{\delta T}{\delta y}(x,0) = 0 \qquad (2.9)$$

The last boundary condition states that the temperature profile is symmetric about x – axis, introducing the dimensionless variables.

$$x = 6\xi^{1}, y = b\eta$$

$$x = b\xi^{1}, y = b\eta \qquad (2.10)$$

equation (2.6) transforms to

$$p_e f(n)\frac{\delta T}{\delta\xi^1}\frac{\delta^2 T}{\delta\eta^2}, 0 \le \eta \le c, \qquad (2.11)$$

$$= \frac{\delta^2 T}{\delta\eta^2} + \frac{4\mu u_{max}^2}{k(1-c)4}(n-c)(\eta - c + \alpha)$$

$$c \in \eta \in 4,$$

where $p_e\left(=\frac{\rho c_v u_{max} b}{k}\right)$ is the Peclet's number

$$f(\eta) = 1, 0 \le \eta \le c$$

$$= \frac{1-\eta^2 - 2c(1-\eta)}{(1-c)^2}, c \le \eta \le 1 \qquad (2.12)$$

and

$$\alpha = \frac{bTy(1-c)^2}{2\mu u_{max}} \qquad (2.13)$$

Defining new dimensionless variables given by

$$\xi^1 = p_e\xi, t = \frac{K(1-c)^4(T-T_0)}{4\mu u_{max}^2} \qquad (2.14)$$

equation (2.11) transforms to

$$f(\eta)\frac{\delta t}{\delta \xi} = \frac{\delta^2 t}{\delta \eta^2} + f_1(\eta) \qquad (2.15)$$

where

$$f_1(\eta) = 0, 0 \le \eta \le c,$$

$$= (\eta - c)(\eta - c + \alpha), c \le \eta \le 1 \qquad (2.16)$$

The boundary conditions (2.7) to (2.9) reduce to

$$t(o,\eta) = 0 \qquad (2.17)$$

$$\frac{\delta t}{\delta n}(\xi, 4) = 0 \qquad (2.18)$$

$$\frac{\delta t}{\delta \eta}(\xi, o) = 0 \qquad (2.19)$$

Equation (2.15) to (2.19) constitute a boundary value problem.

3. MATHEMATICAL SOLUTION

We assume a solution of the form

$$t = t_1 + t_2 \qquad (3.1)$$

where t_1 is an approximate solution for large values of ξ. Since for large longitudinal distances one expects that the initial disturbances in the temperature profile will be damped out and hence the temperature will rise linearly with the distance, we take

$$t_1 = H(\eta) + a\xi \qquad (3.2)$$

Such that $\frac{\delta t_1}{\delta \eta} = 0$ at $\eta = 0$ and $\eta = 1$ and t_1 is a solution of (2.15)

Substitution of (3.2) in (2.15) and solving the resulting equation for $H(\eta)$ under the boundary conditions $\frac{dH}{d\eta} = 0$ at $\eta = 0$ and $\eta = 1$ (which is a direct consequence of $\frac{\delta t_1}{\delta \eta} = 0$ at $\eta = 0$ and $\eta = 1$) and using the fact that $\frac{dH}{d\eta}$ at $\eta = c$ is unique gives the value of to be

$$\alpha = \frac{(1-c)^2\{2(1-c) + 3\alpha\}}{2(c+2)} \qquad (3.3)$$

Thus, the equation to determine $H(\eta)$ are

$$\frac{dH(\eta)}{d\eta} = a\eta, 0 \le \eta \le c,$$

$$= \int [af(\eta) - f_1(\eta)]d\eta, c \le \eta \le 1 \qquad (3.4)$$

Integrating (3.4), we get

$$H(\eta) = \frac{a\eta^2}{2}, 0 \le \eta \le c.$$

$$= Q_1 + Q_2\eta + \left[\frac{a(1-2c)}{2(1-c)^2} - \frac{c(c-\alpha)}{2}\right]\eta^2 + \left[\frac{ac}{3(1-c)^2} + \frac{2c-\alpha}{6}\right]\eta^3$$

$$+ \left[\frac{a}{12(1-c)^2} + \frac{1}{12}\right]\eta^4, c \le \eta \le 1 \qquad (3.5)$$

where

$$Q_1 = \frac{ac}{12(1-c)^2}[a - 12c + 6c^2 - 3c^3] - \frac{c}{12}[2(2+3\alpha) - 12(\alpha+1) + 4c^2(\alpha+3)c^3 \qquad (3.6)$$

$$Q_2 = \frac{a(3c-2)}{3(1-c)^2} + \frac{6c^2 - 6c(\alpha+1) + 3\alpha + 2}{6} \qquad (3.7)$$

From equation (2.15) to (2.19) and (3.1) t_2 must satisfy the boundary conditions $\frac{\delta t_2}{\delta \eta} = 0$ $\eta = 0$ and $\eta = 1$ and $t_2(0,\eta) = -H(\eta)$

By taking

$$t_2(\xi, \eta) = x(\xi)Y(\eta) \qquad (3.8)$$

leads to the following pair of ordinary differential equations.

$$\frac{dx(\xi)}{d\xi} + \lambda^2 x(\xi) = 0 \qquad (3.9)$$

and

$$\frac{d^2Y(\eta)}{\delta\eta^2}+\lambda^2+Y(\eta)=0 \qquad (3.10)$$

where λ^2 is a constant,

Equation (3.9) gives)

$$x(\xi)=Ae^{A^2\xi} \qquad (3.11)$$

Equation (3.10) has to be solved under the boundary conditions.

$$y^1(\eta)=0 \text{ at } \eta=0 \text{ and } \eta=1 \qquad (3.12)$$

Equations (3.10) and (3.12) constitute a Sturm-Liouville problem. Hence there is a set of eigen values of λ^2, say λ^2, λ_2^2...... and a corresponding set of non-sero eigen functions Y_1, Y_2.....satisfying (3.10) and (3.12). The Y_l are orthogonal with respect to the weight function f(□) on the interval $0 \le \eta \le 1$

$$\text{i.e.} \int_0^1 f(\eta)\gamma_i(\eta)\gamma_j(\eta)d\eta=0 \text{ if } i\neq j \qquad (3.13)$$

Thus

$$t_2(\xi,\eta)=\Sigma A_n e^{-\lambda_n^2\xi}Y_n(\eta) \qquad (3.14)$$

4. RESULTS AND DISCUSSIONS

For the calculation of eigen values, eigen functions and expansion coefficients, We have used Rayleigh-Ritz-method.

We have taken the approximating functions satisfying the boundary conditions (3.12) as

$$Y(\eta,\alpha_1,\alpha_2;.....\alpha_n)=\sum_{r=1}^{n}\alpha r\cos\pi r\eta \qquad (4.1)$$

$$\frac{\delta Y}{\delta\eta}=-\pi\sum_{r=1}^{n} r d r\sin\pi r\eta \qquad (4.2)$$

The equations to determine r are

$$\frac{\delta J}{\delta\alpha_r}=0,\ r=1,2,3,......n, \qquad (4.3)$$

where

$$J=\left[\int_0^1 Y'^2 d\eta-\lambda_n^2\left\{\int_0^c Y^2 d\eta+\int_c^1\frac{1-\eta^2-2c(1-\eta)y^2 d\eta}{(1-c)^2}\right\}\right] \qquad (4.4)$$

Substituting (4.1), (4.2) in (4.4) and simplifying, we get

$$J=\sum_{r=1}^{n}\frac{r^2\pi^2}{2}\alpha_r^2-\lambda_n^2\left\{\sum_{r=1}^{n}\alpha_r^2 p(r)+\Sigma\alpha_i\alpha_j Q(i,j)\right\} \qquad (4.5)$$

$$i=1,2,......n-1$$

$$j=i+1,i+2,.....n$$

where

$$p(r)=\frac{1}{2}-\frac{c^2}{(1-c)^2}B_1(r)+\frac{2c}{(1-c)^2}B_3(r)-B_5(r) \qquad (4.6)$$

$$Q(i,j)=\frac{c^2}{(1-c)^2}B_2(i,j)+\frac{2c}{(1-c)^2}B_4(i,j)-B_6(i,j) \qquad (4.7)$$

$$B_1(r)=\frac{1-c}{2}-\frac{\sin 2\pi rc}{4\pi r} \qquad (4.8)$$

$$B_2(i,j)=\frac{1}{\pi}\left[\frac{\sin(i+j)\pi c}{(i+j)}+\frac{\sin(i-j)\pi c}{(i-j)}\right] \qquad (4.9)$$

$$B_3(r)=\frac{1-c^2}{4}\frac{-c\sin 2\pi rc}{4\pi r}+\frac{1-c\cos 2\pi rc}{8\pi^2 r^2} \qquad (4.10)$$

$$B_4(i,j)\frac{(-1)^{i+j}2(i^2+j^2)}{\pi^2(i^2-j^2)}-\frac{c}{\pi}\left[\frac{\sin\pi(i+j)c}{(i+j)}+\frac{\sin\pi(i-j)c}{(i-j)}\right]-\frac{1}{\pi^2}$$
$$\left[\frac{\cos\pi(i+j)c}{(i+j)^2}+\frac{\cos\pi(i-j)c}{(i-j)^2}\right] \qquad (4.11)$$

$$B_5(r)=\frac{1-c^3}{6}-\frac{c^2\sin 2\pi rc}{4\pi r}+\frac{1-c\cos 2\pi rc}{4\pi^2 r^2}+\frac{\sin 2\pi rc}{8\pi^3 r^3} \qquad (4.12)$$

and

$$B_6(i,j)=\frac{(-1)^{i+j}4(i^2+j^2)}{\pi^2(i-j)^2}-\frac{c^2}{\pi}\left[\frac{\sin\pi(i+j)c}{(i+j)}+\frac{\sin\pi(i-j)c}{(i-j)}\right]$$
$$-\frac{2c}{\pi^2}\left[\frac{\cos\pi(i+j)c}{(i+j)^2}+\frac{\cos\pi(i-j)c}{(i-j)^2}\right]+\frac{2}{\pi^3}\left[\frac{\sin\pi(i+j)c}{(i+j)^3}+\frac{\sin\pi(i-j)c}{(i-j)^3}\right] \qquad (4.13)$$

Equation (4.3) are a system of homogenous equations in unknown r. For this system to have a nontrivial solution.

$$|a_{ij}-\lambda_n^2 b_{ij}|=0 \qquad (4.14)$$

where

$$a_{ij}=r^2\pi^2 \text{ when } i=j=r$$
$$=0 \text{ when } i\neq j, \qquad (4.15)$$

$$b_{ij}=\rho(r) \text{ when } i=j=r,$$
$$=Q(i,j) \text{ when } i\neq j \qquad (4.16)$$

The values of λn which give a nontrivial solution are the eigen values and then the corresponding $\square_i$ are obtained by solving the system of equations (4.3). Knowing λ_1, α_i we can calculate the eigen values from (4.1). There after using (4.16), the expansion coefficients A_i can be evaluated. We have calculated the first four eigen values and expansion coefficients A_i for various values of c. The coefficients A_i have been calculated by taking the first eigen function, corresponding to zero eigen value, to be unity. Eigen functions are evaluated with the assumptions that $Y_n(0)$ = 1.0. The results are given in table 1 as well as 2.

Table - 1

Eigen values and Expansion Coefficients:

C	i	λ_i^2	A_i
	1	0.0	–0.137857
	2	15.32	0.000220
0.0	3	66.54	0.000035
	4	162.47	0.00023
	1	0.00	–0.070224
0.1	2	13.13	0.000133
	3	56.63	0.000017
	4	124.14	0.000884
0.25	1	0.00	–0.028056
	2	10.10	0.000013
	3	42.02	0.000119
	4	88.73	0.000046
	1	0.00	0.009381
0.5	2	5.37	0.000215
	3	2.43	0.000814
	4	62.62	0.000063

Table – 2

Eigen functions are calculated and tabulated as given below:

	n	Y_2	Y_3	Y_4
0	0.0	1.0000	1.0000	1.0000
	0.2	0.9000	0.3624	-1.9392
	0.4	0.4853	-0.4846	-1.2537
0.0	0.6	-0.2624	-0.4239	4.3572
	0.8	-0.9853	-0.0154	0.7537
	1.0	-1.2755	0.1230	-4.8362
	0.0	1.0000	1.0000	1.0000
0.1	0.2	0.8827	0.3775	-0.7547
	0.4	0.4532	-0.5100	-0.8690
	0.8	-0.2685	-0.6060	1.8788
	0.6	-0.9532	0.0109	-1.4400
	1.0	-1.2283	0.2569	-2.2482
	0.0	1.0000	1.0000	1.0000
	0.2	0.8566	0.4109	0.9579
0.25	0.4	0.4055	-0.5601	0.2109
	0.6	-0.2774	-0.6731	-0.8576
	0.8	-0.9055	0.0601	-0.7109
	1.0	-1.1584	0.5243	-0.2006
	0.0	1.0000	1.0000	1.0000
	0.2	0.8001	0.4962	-0.1416
0.5	0.4	0.3001	-0.3156	-0.8650
	0.6	-0.2999	-1.3096	0.2801
	0.8	-0.8001	0.3156	0.3650
	1.0	-1.0002	1.6270	-0.2770

5. CONCLUSIONS

(i) The graphs for the temperature $t(\xi, \eta)$ versus η for fixed values of ξ and c are drawn. They are shown in fig. 1-2.

(ii) It is concluded from the graph that $t(\xi, \eta)$ decreases with the increase of c. Hence it decreases with the increase of yield stress.

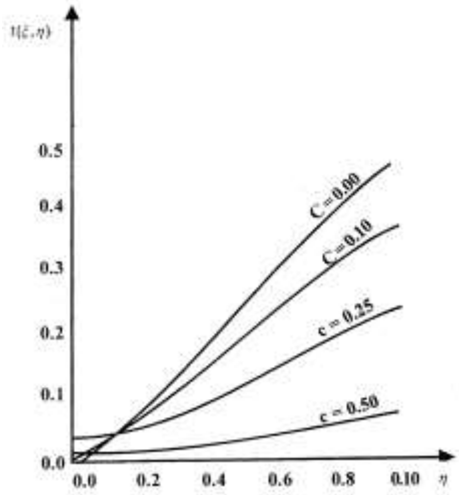

Fig. 1: TEMPERATURE PROFILES (ξ = 0.25)

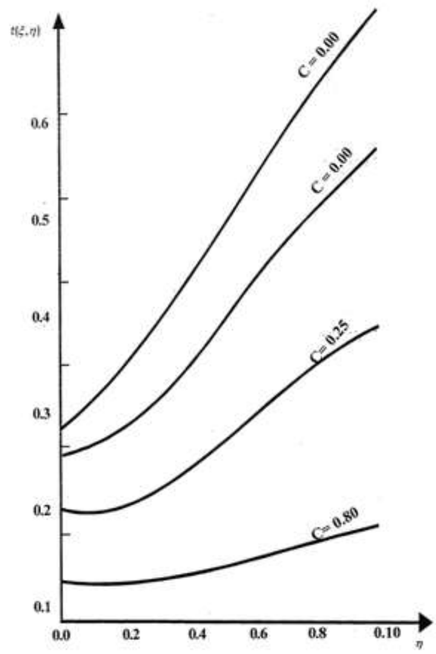

Fig. 2: TEMPERATURE PROFILES (ξ = 0.50)

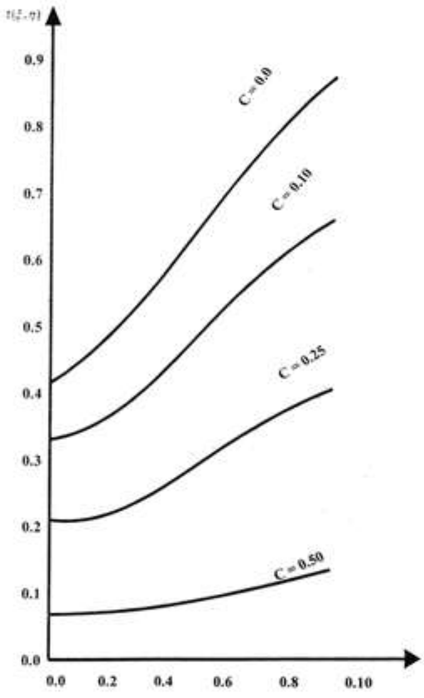

Fig. 3: TEMPERATURE PROFILES (ξ = 0.75)

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