

Some Fixed Point Results in Menger Space Using the Notion of CLR and JCLR Property

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Abstract – The point of this paper is to demonstrate, basically, three normal fixed point hypotheses for six self mappings of a Menger space utilizing two feebly viable sets having CLR/JCLR-property and fulfilling a certain connection. These sum up a few realized outcomes including those of Kohli et. al.

Keywords: Menger Space, CLR property and JCLR property.

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INTRODUCTION

As of now, an intriguing territory of exploration is demonstrating brings about Menger space. They presented the idea of probabilistic Menger space. They demonstrated fascinating outcomes with regards to summed up the consequences of Kumar and Pant. They presented the idea of CLR-property and this is additionally summed up as JCLR-property. Utilizing these ideas, we summed up the previously mentioned results. We saw that the states of closedness of the subspaces and coherence of the mappings are not required in building up our outcomes.

As normal $\mathbb{R}$ represents the arrangement of every single genuine number, $\mathbb{R}^+$ represents the arrangement of all non-negative genuine numbers and $\mathbb{N}$ represents the arrangement of every regular number.

Preliminaries

We hereunder give the accompanying definitions and the outcome needed in ensuing area.

Definition 2.1. ([2]) A mapping $F : \mathbb{R} \rightarrow \mathbb{R}^+$ is called a distribution if and only if it is nondecreasing, left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$. The set of all distribution functions are denoted by L .

For example, Heaviside function $H : \mathbb{R} \rightarrow \mathbb{R}^+$, defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0 \end{cases}$$

is a distribution function.

Definition 2.2. ([2]) Probabilistic metric space (PM-space) is an ordered pair (X, F) , where X is a non empty set and $F : X \times X \rightarrow L$ is defined by $(p, q) \rightarrow F_{p,q}$ where $\{F_{p,q} : p, q \in X\} \subseteq L$, and the functions $F_{p,q}$ satisfy the following:

- (a) $F_{p,q}(t) = 1$ for all $t > 0$ if and only if $p = q$;
- (b) $F_{p,q}(0) = 0$;
- (c) $F_{p,q}(t) = F_{q,p}(t)$;
- (d) $F_{p,q}(t) = 1$ and $F_{q,r}(s) = 1$, then $F_{p,r}(t + s) = 1$.

Definition 2.3. ([2]) A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a triangular norm (or t-norm) if

- (a) $T(0, 0) = 0$ and $T(a, 1) = a$ for all $a \in [0, 1]$;
- (b) $T(a, b) = T(b, a)$, for all $a, b \in [0, 1]$;
- (c) $T(a, b) \supseteq T(c, d)$ for all $a, b, c, d \in [0, 1]$ with $a \supseteq c$ and $b \supseteq d$;

(d) $T(T(a, b), c) = T(a, T(b, c))$ for all $a, b, c, d \in [0, 1]$.

DEFINITION 2.4. ([2]) A Menger space is a triplet (X, F, T) ,

where (X, F) is a Probabilistic metric space and T is a t-norm such that for all $p, q, r \in X$ and all $t, s > 0$,

$$F_{p,r}(s+t) > T(F_{p,q}(s), F_{q,r}(t)).$$

DEFINITION 2.5. ([9]) Self mappings f and g of a Menger space (X, F, T) are said to be weakly compatible if and only if for any $t > 0$, $F_{fx,gx}(t) = 1$ for some $x \in X$ implies $F_{fgx,gfx}(t) = 1$; i.e. $fx = gx$ for some $x \in X$ implies $fgx = gfx$.

DEFINITION 2.6. ([1]) A function $\phi : (\mathbb{R}^+)^4 \rightarrow \mathbb{R}$ is said to be an implicit relation if

- (i) ϕ is consistent,
- (ii) ϕ is Monotonic expanding in the main contention and
- (iii.) ϕ fulfills the accompanying conditions:

for $x, y > 0$, $\phi(x, y, x, y) > 0$ or $\phi(x, y, y, x) > 0$ implies $x > y$,

- (a) $\phi(x, x, 1, 1) > 0$ implies $x > 1$.

EXAMPLE 2.1. Define $\phi : (\mathbb{R}^+)^4 \rightarrow \mathbb{R}$ by $\phi(x_1, x_2, x_3, x_4) = ax_1 + bx_2 + cx_3 + dx_4$ with $a + b + c + d = 0, a + b > 0, a + c > 0$ and $a + d > 0$. Clearly, ϕ is an implicit relation. In particular,

- (i) $\phi(x_1, x_2, x_3, x_4) = 6x_1 - 3x_2 - 2x_3 - x_4$,
- (ii) $\phi(x_1, x_2, x_3, x_4) = 5x_1 - 3x_3 - 2x_4$ are implicit relations.

Notation: Let Φ be the class of all implicit relations.

DEFINITION 2.7. ([8]) Let (X, F, T) be a Menger space, where T is continuous t-norm.

1. A sequence $\{p_n\}$ in X is said to converge to a point p in X (written as $p_n \rightarrow p$) if for every $\epsilon \in (0, 1)$ and $\lambda > 0$, there exists a positive integer $M(\epsilon, \lambda)$ such that $F_{p_n,p}(\epsilon) > 1 - \lambda$ for all $n > M(\epsilon, \lambda)$.
2. A sequence $\{p_n\}$ in X is said to be Cauchy if for each every $\epsilon \in (0, 1)$ and $\lambda > 0$, there is a positive integer $M(\epsilon, \lambda)$ such that $F_{p_n,p_m}(\epsilon) > 1 - \lambda$ for all $n, m \in \mathbb{N}$ with $n, m > M(\epsilon, \lambda)$.
3. A Menger space (X, F, T) is said to be complete if every Cauchy sequence in X converges to a point of it.

LEMMA 2.1 ([9]). Let $(X, F, *)$ be a sequence in a Menger space (X, F, T) . If there is a $k \in (0, 1)$ such that

$$F_{x,y}(kt) > F_{x,y}(t)$$

for all $x, y \in X$ and $t > 0$, then $y = x$.

DEFINITION 2.8. Let (X, F, T) be a Menger space, where T denotes a continuous t-norm and f, g, h, k be self-mappings on X .

(a) The ordered pairs (f, g) and (h, k) are said to satisfy the "common limit in the range of g " (CLR_g -) property if and only if there exist sequences

$\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} F_{fx_n, gx}(t) = \lim_{n \rightarrow \infty} F_{gx_n, gx}(t) = \lim_{n \rightarrow \infty} F_{hx_n, gx}(t) = \lim_{n \rightarrow \infty} F_{kx_n, gx}(t)$$

for some $x \in X$ and for all $t > 0$.

(b) The ordered pairs (f, g) and (h, k) are said to satisfy the "joint common limit in the ranges of g and k " ($JCLR_{gh}$ -) property if and only if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $ku = gu$ and

$$\lim_{n \rightarrow \infty} F_{fx_n, gu}(t) = \lim_{n \rightarrow \infty} F_{gx_n, gu}(t) = \lim_{n \rightarrow \infty} F_{hx_n, gu}(t) = \lim_{n \rightarrow \infty} F_{kx_n, gu}(t)$$

for some $u \in X$ and for all $t > 0$.

Main theorem

THEOREM 3.1. Let (X, F, T) be a Menger space, where T denotes a continuous t-norm and f, g, h, k, p and q be self-mappings of X , satisfying:

1. $p(X) \subseteq fg(X)$ and $q(X) \subseteq hk(X)$;
2. the pairs $\{p, hk\}$ and $\{q, fg\}$ be weakly compatible;
3. the ordered pairs (p, hk) and (q, fg) share either
4. CLR_p -property or CLR_q -property;
5. $\phi(F_{px, qy}(\alpha t), F_{hky, fgy}(t), F_{px, hky}(t), F_{qy, fgy}(\alpha t)) > 0$, for all $x, y \in X$ & $t > 0$ and for some $\phi \in \Phi$ & $\alpha \in (0, 1)$;
6. h commutes with k and 'either p commutes with h or with k ';
7. f commutes with g and 'either q commutes with f or with g '. Then f, g, h, k, p and q have a unique common fixed point in X .

Proof. Case I: Suppose (iii)(a) holds.

By definition, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} px_n = \lim_{n \rightarrow \infty} h k x_n = \lim_{n \rightarrow \infty} q y_n = \lim_{n \rightarrow \infty} f g y_n = pu, \text{ for some } u \in X.$$

Since $p(X) fg(X)$, there is a $v \in X$ such that $pu = fgv$. By taking $x = x_n$ and

$y = v$ in (iv), we get that

$$\phi(F_{px,n,qv}(\alpha t), F_{h k x,n,fgv}(t), F_{px,n,h k x n}(t), F_{qv,fgv}(\alpha t)) > 0.$$

As $n \rightarrow \infty$, the above becomes

$$\phi(F_{pu,qv}(\alpha t), F_{pu,fgv}(t), F_{pu,pu}(t), F_{qv,fgv}(\alpha t)) > 0.$$

So, by the property of ϕ , $F_{pu,qv}(\alpha t) \preceq F_{pu,qv}(t)$. By Lemma(2.1), $pu = qv$. Since $q(X) \subseteq hk(X)$, there is a $w \in X$ such that $qv = hkw$. By taking $x = w$ and $y = v$ in (iv), we get that

$$\phi(F_{pw,qv}(\alpha t), F_{hkw,fgv}(t), F_{pw,hkw}(t), F_{qv,fgv}(\alpha t)) > 0$$

$$\text{i.e., } \phi(F_{pw,qv=hkw}(\alpha t), 1, F_{pw,hkw}(t), 1) > 0$$

o, by the property of ϕ , we have

$$\phi(F_{pw,qv=hkw}(t), 1, F_{pw,hkw}(t), 1) \preceq 0 \Rightarrow F_{pw,hkw}(t) \preceq 1 \Rightarrow pw = hkw.$$

Thus $pw = hkw = pu = fgv = qu = z$ (say).

Since p, hk and q, fg are weakly compatible, we have $p(hk)w = hk(p)w$

and $q(fg)v = fg(q)v$. i.e., $pz = hkz$ and $qz = fgz$.

By putting $x = z$ and $y = v$ in (iv), we get that

$$\phi(F_{pz,qv=z}(\alpha t), F_{h k z,pz,fgv=z}(t), F_{pz,h k z=pz}(t), F_{qv=z,fgv=z}(\alpha t)) > 0$$

i.e.,

$$\phi(F_{pz,z}(\alpha t), F_{pz,z}(t), 1, 1) > 0 \Rightarrow \phi(F_{pz,z}(t), F_{pz,z}(t), 1, 1) > 0$$

$$\Rightarrow F_{pz,z}(t) \preceq 1 \Rightarrow pz = z.$$

Similarly, by taking $x = w$ and $y = z$ in (iv), we get that $qz = z$. Thus

$$pz = hkz = z = qz = fgz.$$

Since h commutes with k , we have $hk(hz) = h(hkz) = hz$. Suppose p commutes with h , so $p(hz) = h(pz) = hz$; by taking $x = hz$ and $y = z$ in (iv), we get that $\phi(F_{phz=hz,qz=z}(\alpha t), F_{h k hz=hz,fgz=z}(t), F_{phz=hz,h k hz=hz}(t), F_{qz=z,fgz=z}(\alpha t)) > 0$

$$\Rightarrow \phi(F_{hz,z}(\alpha t), F_{hz,z}(t), F_{hz,hz}(t), F_{z,z}(\alpha t)) > 0$$

$$\Rightarrow \phi(F_{hz,z}(t), F_{hz,z}(t), 1, 1) > 0$$

$$\Rightarrow F_{hz,z}(t) > 1 \Rightarrow hz = z.$$

Since $h k z = z$, follows that $kz = z$. Thus $hz = kz = pz = z$. Suppose p commutes with k , so $p(kz) = k(pz) = kz$. Since h commutes with k , we have $hk(kz) = k(hkz) = kz$. By taking $x = kz$ and $y = z$ in (iv), we get that $kz = z$. Since $h k z = z$, follows that $hz = z$. Thus $hz = kz = pz = z$.

Now, (vi) is similar to (v) when p, h, k are replaced by q, f, g respectively. Hence as above, we get $z = fz = gz = qz$. Thus $fz = gz = hz = kz = pz = qz = z$. Hence z is a common fixed point of f, g, h, k, p and q .

Case II: Suppose (iii)(b) holds. By definition, there exist sequences $\{x_n\}$ and

$\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} px_n = \lim_{n \rightarrow \infty} h k x_n = \lim_{n \rightarrow \infty} q y_n = \lim_{n \rightarrow \infty} f g y_n = qv$$

for some $v \in X$. Since $q(X) \subseteq hk(X)$, there is a $u \in X$ such that $qv = hku$. By taking $x = u$ and $y = y_n$ in (iv), we get that $qv = pu$. Since $p(X) fg(X)$, there is a $w \in X$ such that $pu = fgw$. By taking $x = u$ and $y = w$ in (iv), we get that $qw = fgw$. Thus $pu = hku = qv = fgw = qw = z$ (say). Since (p, hk) and (q, fg) are weakly compatible, $p(hk)u = hk(p)u$ and $q(fg)w = fg(q)w$. i.e., $pz = hkz$ and $qz = fgz$. From this stage, the proof is the same given in the previous case. Thus, z is a common fixed point of f, g, h, k, p and q .

Uniqueness: If w is also a common fixed point of f, g, h, k, p and q . By taking $x = z$ and $y = w$ in (iv), we get that

$$\phi(F_{pz,qw}(\alpha t), F_{h k z,fgw}(t), F_{pz,h k z}(t), F_{qw,fgw}(\alpha t)) > 0$$

$$\text{i.e., } \phi(F_{z,w}(\alpha t), F_{z,w}(t), F_{z,z}(t), F_{w,w}(\alpha t)) > 0$$

So, by property of ϕ , $F_{z,w}(\alpha t) \preceq F_{z,w}(t)$. By lemma(2.1), we get that $w = z$. Hence z is the unique common fixed point of f, g, h, k, p and q . This completes the proof of the theorem.

NOTE 3.1. Theorem (3.1) is also valid if

$$(i) \quad (iv) \text{ is replaced by } \phi(F_{px,qv}(\alpha t), F_{h k x,fgv}(t), F_{px,h k x}(\alpha t), F_{qv,fgv}(t)) > 0.$$

$$(ii) \quad (i) \text{ is replaced by } q(X) \subseteq hk(X) \text{ and } (iii) \text{ is replaced by } (p, hk) \text{ and } (q, fg) \text{ share } CLR_{(fg)}\text{-property.}$$

- (iii) (i) is replaced by $p(X) \subseteq fg(X)$ and (iii) is replaced by (p, hk) and (q, fg) share $CLR_{(hk)}$ -property.

Now we give the following example in support of our Theorem (3.1).

EXAMPLE 3.1. Let $X = [0, \infty)$, $a \in b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and

$$F_{x,y}(t) = \frac{t}{t+|x-y|}$$

for all $x, y \in X$ and for all $t > 0$. Then $(X, F, \in)$ is a Menger space. Define self mappings f, g, h, k, p and q on X by $fx = x$, $gx = x/2$,

$$hx = x^3, kx = x,$$

$$p(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ \frac{1}{2} & \text{if } x > 1, \end{cases}$$

$qx = 0$, for all $x \in X$. Define $\phi : (R^+)^4 \rightarrow R$ by

$$\phi(x_1, x_2, x_3, x_4) = 6x_1 - 3x_2 - 2x_3 - x_4.$$

Then ϕ is an implicit relation.

For $x \in [0, 1]$ and $y \in X$, we have

$$\begin{aligned} \phi(F_{0,0}(\alpha t), F_{x,3}(y(t)), F_{0,x^3}(t), F_{0,y}(\alpha t)) \\ = 6 - 3\frac{t}{t+|x^3-y|} - 2\frac{t}{t+x^3} - \frac{\alpha t}{\alpha t+y} \end{aligned}$$

$$> 6 - 3 - 2 - 1 = 0.$$

For $x > 1$ and $y \in X$, we have

$$\begin{aligned} \phi(F_{\frac{1}{2},0}(\alpha t), F_{x^3,y}(t), F_{\frac{1}{2},x^3}(t), F_{0,y}(\alpha t)) \\ = 6\frac{\alpha t}{\alpha t+\frac{1}{2}} - 3\frac{t}{t+|x^3-y|} - 2\frac{t}{t+x^3-\frac{1}{2}} - \frac{\alpha t}{\alpha t+y} \end{aligned}$$

$$6 - 3 - 2 - 1 = 0.$$

Different states of the Theorem are inconsequentially fulfilled. Unmistakably '0' is the remarkable regular fixed purpose of f, g, h, k, p and q in X .

Presently, taking $g = k = I$ (the character planning on X) in Theorem (3.1), we have the accompanying:

COROLLARY 3.1. Let (X, F, T) be a Menger space, where T denotes a continuous t -norm and f, h, p and q be self-mappings of X , satisfying:

- (i) $p(X) \subseteq f(X)$ and $q(X) \subseteq h(X)$;
(ii) the pairs $\{p, h\}$ and $\{q, f\}$ are weakly compatible;

- (iii) the ordered pairs (p, h) and (q, f) share either (a) CLR_p -property or (b) CLR_q -property;

$$(iv) \quad \phi(F_{px,qy}(\alpha t), F_{hx,fy}(t), F_{px,hx}(t), F_{qy,fy}(\alpha t)) > 0,$$

for all $x, y \in X$ & $t > 0$ and for some $\phi \in \Phi$ & $\alpha \in (0, 1)$. Then f, g, h, k, p and q have a unique common fixed point in X .

Now, we prove the following:

THEOREM 3.2. Let (X, F, T) be a Menger space, where T denotes a continuous t -norm and f, g, h, k, p and q be self-mappings of X , satisfying:

- (i) the pairs $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible;
(ii) the ordered pairs (p, hk) and (q, fg) share $JCLR_{(hk)(fg)}$ -property;
(iii) $\phi(F_{px,qy}(\alpha t), F_{hxx,fgy}(t), F_{px,hxx}(t), F_{qy,fgy}(\alpha t)) > 0$, for all $x, y \in X$ & $t > 0$ and for some $\phi \in \Phi$ & $\alpha \in (0, 1)$;
(iv) h commutes with k and 'either p commutes with h or with k '; f commutes with g and 'either q commutes with f or with g '. Then f, g, h, k, p and q have a unique common fixed point in X .

Proof. Suppose (p, hk) and (q, fg) share $JCLR_{(hk)(fg)}$ -property, by definition, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that for some $u \in X$. By taking $x = u$ and $y = y_n$ in (iii), we get that

$$\phi(F_{pu,qyn}(\alpha t), F_{hku,fgy_n}(t), F_{pu,hku}(t), F_{qyn,fgy_n}(\alpha t)) > 0 \text{ As } n \rightarrow \infty, \text{ we get that}$$

$$\phi(F_{pu,hku}(\alpha t), F_{hku,hku}(t), F_{pu,hku}(t), F_{hku,hku}(\alpha t)) > 0$$

i.e.,

$$\phi(F_{pu,hku}(\alpha t), 1, F_{pu,hku}(t), 1) > 0$$

$$\Rightarrow \phi(F_{pu,hku}(t), 1, F_{pu,hku}(t), 1) > 0$$

$$\Rightarrow F_{pu,hku}(t) \text{ " } F_{pu,hku}(t) \Rightarrow pu = hku \text{ (by lemma(2.1)).}$$

By taking $x = x_n$ and $y = u$ in (iii), we get that

$$\phi(F_{pxn,qu}(\alpha t), F_{hxxn,fgu}(t), F_{pxn,hxxn}(t), F_{qu,fgu}(\alpha t)) > 0$$

As $n \rightarrow \infty$, we get that

$$\phi(F_{fgu,qu}(\alpha t), F_{fgu,fgu}(t), F_{fgu,fgu}(t), F_{qu,fgu}(\alpha t)) > 0$$

$$\Rightarrow F_{fgu,qu}(\alpha t) \text{ " } 1 \Rightarrow fgu = qu.$$

Thus $fgu = qu = pu = hku = z$ (say). Since p, hk and q, fg are weakly compatible, $p(hk)u = hk(p)u$ and $q(fg)u = fg(q)u$. i.e, $pz = hkz$ and $qz = fgz$. From this stage, the verification is a similar given in the Theorem (3.1). Subsequently, we get that z is a typical fixed purpose of f, g, h, k, p and q .

Uniqueness follows inconsequentially.

Note 3.2. Theorem (3.2) is also valid if (iii) is replaced by

$\phi(F_{px,qy}(\alpha t), F_{hgx,fgy}(t), F_{px,hgx}(\alpha t), F_{qy,fgy}(t)) > 0$. We now give the following example in support of Theorem (3.2).

EXAMPLE 3.2. Let $X = [0, \infty)$, $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and

$F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and for all $t > 0$. Then $(X, F, *)$ is a Menger

space.

Define self mappings f, g, h, k, p and q on X by $fx = x, gx = x^2,$

$hx = x^5, kx = x,$

$$p(x) = \begin{cases} 0 & \text{if } x \leq 3, \\ 2 & \text{if } x > 3, \end{cases}$$

$qx = 0$, for all $x \in X$. Define $\phi : (R+)^4 \rightarrow R$ by $\phi(x_1, x_2, x_3, x_4) = 5x_1 - 3x_2 - 2x_4$. Then ϕ is an implicit relation.

For $x \leq 3$ and $y \in X$, we have

$$\phi(F_{0,0}(\alpha t), F_{x^5,y^2}(t), F_{0,x^5}(t), F_{0,y^2}(\alpha t)) = 5 - 3\frac{t}{t+|x^5-y^2|} - \frac{\alpha t}{\alpha t+y^2}$$

$$> 5 - 3 - 2 = 0.$$

For $x > 3$ and $y \in X$, we have

$$\phi(F_{2,0}(\alpha t), F_{x^5,y^2}(t), F_{2,x^5}(t), F_{0,y^2}(\alpha t)) = 5\frac{\alpha t}{\alpha t+2} - 3\frac{t}{t+|x^5-y^2|} \lim_{n \rightarrow \infty} px_n = \lim_{n \rightarrow \infty} hkn_n = \lim_{n \rightarrow \infty} qy_n = \lim_{n \rightarrow \infty} fgy_n = pu = 0$$

$$> -3 - 2 = 0.$$

Different states of the Theorem are inconsequentially fulfilled. Unmistakably '0' is the one of a kind normal fixed purpose of f, g, h, k, p and q in X .

By taking $g = k = I$ (the personality planning on X) in Theorem (3.2), we have the accompanying:

Corollary 3.2. Let (X, F, T) be a Menger space, where T means a continuous t -standard and f, h, p and q act naturally mappings of X , fulfilling:

- (i) the pairs $\{p, h\}$ and $\{q, f\}$ are weakly compatible;
- (ii) the ordered pairs (p, h) and (q, f) share $JCLR_{hf}$ -property;
- (iii) $\phi(F_{px,qy}(\alpha t), F_{hgx,fgy}(t), F_{px,hgx}(t), F_{qy,fgy}(\alpha t)) > 0$,

for all $x, y \in X$ & $t > 0$ and for some $\phi \in \Phi$ & $\alpha \in (0, 1)$.

At that point f, h, p and q have a one of a kind basic fixed point in X .

Presently we demonstrate the accompanying:

THEOREM 3.3. Let (X, F, T) be a Menger space, where T means a ceaseless t -standard and f, g, h, k, p and q act naturally mappings of X , fulfilling:

- (i) the sets $\{p, hk\}$ and $\{q, fg\}$ are feebly viable;
- (ii) the requested sets (p, hk) and (q, fg) share $JCLR_{pq}$ -property;
- (iii) one of the accompanying holds: all things considered

$$(a) \phi(F_{hgx,fgy}(\alpha t), F_{px,fgy}(t), F_{hgx,qy}(t), F_{px,qy}(\alpha t)) > 0,$$

or (b) $\phi(F_{hgx,fgy}(\alpha t), F_{px,fgy}(t), F_{hgx,qy}(\alpha t), F_{px,qy}(t)) > 0$, for all $x, y \in X$ & $t > 0$ and for some $\phi \in \Phi$ & $\alpha \in (0, 1)$;

- (i) h drives with k and 'either p drives with h or with k ';
- (ii) f drives with g and 'either q drives with f or with g '. At that point f, g, h, k, p and q have a novel regular fixed point in X .

Proof. Since (p, hk) and (q, fg) share $JCLR_{pq}$ -property, by definition, there exist arrangements $\{x_n\}$ and $\{y_n\}$ in X such that

for some $u \in X$.

Case I: Assume (iii)(a) holds. By taking $x = x_n$ and $y = u$ in (iii)(a), we

get that $\phi(F_{hkn,fgu}(\alpha t), F_{pxn,fgu}(t), F_{hkn,qu}(t), F_{pxn,qu}(\alpha t)) > 0$.

As $n \rightarrow \infty$, we get that

$$\phi(F_{pu=qu,fgu}(\alpha t), F_{pu=qu,fgu}(t), F_{pu=qu,qu}(t), F_{pu=qu,qu}(\alpha t)) > 0$$

i.e,

$$\phi(F_{qu,fgu}(\alpha t), F_{qu,fgu}(t), 1, 1) > 0$$

$$\Rightarrow \phi(F_{qu,fgu}(t), F_{qu,fgu}(t), 1, 1) > 0$$

$$\Rightarrow F_{qu,fgu}(t) > 1 \Rightarrow qu = fgu$$

By taking $x = u$ and $y = y_n$ in (iii)(a), we get that

$$\phi(F_{hku,fgyn}(\alpha t), F_{pu,fgyn}(t), F_{hku,qyn}(t), F_{pu,qyn}(\alpha t)) > 0.$$

As $n \rightarrow \infty$, we get that

$$\phi(F_{hku,pu=qu}(\alpha t), F_{pu,pu=qu}(t), F_{hku,pu=qu}(t), F_{pu,pu=qu}(\alpha t)) > 0$$

i.e,

$$\phi(F_{hku,pu}(\alpha t), F_{pu,pu}(t), F_{hku,pu}(t), F_{pu,pu}(\alpha t)) > 0$$

$$\Rightarrow \phi(F_{hku,pu}(\alpha t), 1, F_{hku,pu}(t), 1) > 0$$

$$\Rightarrow \phi(F_{hku,pu}(t), 1, F_{hku,pu}(t), 1) > 0 \text{ (by the property of } \phi)$$

$$\Rightarrow F_{hku,pu}(t) > 1 \Rightarrow hku = pu.$$

Thus $fgu = qu = hku = pu = z$ (say)

Since $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible,

$$p(hk)u = hk(p)u \text{ and } q(fg)u = fg(q)u$$

i.e, $pz = hkz$ and $qz = fgz$.

From this stage, the confirmation is a similar given in the Theorem(3.1). Consequently, we get that z is a typical fixed purpose of f, g, h, k, p and q .

Case II: Suppose (iii)(b) holds: By taking $x = x_n$ and $y = u$ in (iii)(b), we get that $\phi(F_{hkn,fgu}(\alpha t), F_{pxn,fgu}(t), F_{hkn,qu}(\alpha t), F_{pxn,qu}(t)) > 0$.

As $n \rightarrow \infty$, we get that

$$\phi(F_{pu=qu,fgu}(\alpha t), F_{pu=qu,fgu}(t), F_{pu=qu,qu}(\alpha t), F_{pu=qu,qu}(t)) > 0$$

i.e,

$$\phi(F_{qu,fgu}(\alpha t), F_{qu,fgu}(t), 1, 1) > 0$$

$$\Rightarrow \phi(F_{qu,fgu}(t), F_{qu,fgu}(t), 1, 1) > 0$$

$$\Rightarrow F_{qu,fgu}(t) > 1 \Rightarrow qu = fgu.$$

By taking $x = u$ and $y = y_n$ in (iii)(b), we get that

$$\phi(F_{hku,fgyn}(\alpha t), F_{pu,fgyn}(t), F_{hku,qyn}(\alpha t), F_{pu,qyn}(t)) > 0.$$

As $n \rightarrow \infty$, we get that

$$\phi(F_{hku,pu=qu}(\alpha t), F_{pu,pu=qu}(t), F_{hku,pu=qu}(\alpha t), F_{pu,pu=qu}(t)) > 0$$

i.e,

$$\phi(F_{hku,pu}(\alpha t), F_{pu,pu}(t), F_{hku,pu}(\alpha t), F_{pu,pu}(t)) > 0$$

$$\Rightarrow \phi(F_{hku,pu}(\alpha t), 1, F_{hku,pu}(\alpha t), 1) > 0$$

$$\Rightarrow F_{hku,pu}(\alpha t) = 1 \text{ (by the property of } \phi)$$

$$\Rightarrow hku = pu.$$

Thus $fgu = qu = hku = pu = z$ (say).

Since $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible,

$$p(hk)u = hk(p)u \text{ and } q(fg)u = fg(q)u$$

i.e, $pz = hkz$ and $qz = fgz$. From this stage, the evidence is a similar given in the Theorem(3.1). Subsequently, we get that z is a typical fixed purpose of f, g, h, k, p and q . Uniqueness follows inconsequentially.

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