A Study of Deviating Arguments in Existence Differentialequations of Solutions

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Abstract - Mathematical equations involving derivatives and integrals are used to describe the most *fascinating natural phenomenon. In either scenario, these equations are classified as differential or integral equations. Numerous linear and nonlinear differential equations arise in various fields of physical, biological, social, and engineering science. If, when investigating a system, we discover a differential equation, this is referred to as differential equation modeling of the system. Assumption is made that the system and its subsystems interact instantaneously and there is no delay between them in ODE models. Realistic models, on the other hand, incorporate a small bit of lag. As a result, in order to predict the future, it is necessary to take into account the present and the past, as well as derivatives of the former. Functional differential equations are used to model these models (FDEs). In many cases, FDEs are preferable than ODEs because of the implicit assumption that the system's past influences its present state. The simplest versions of FDEs are known as delay differential equations (DDE). For this reason, they are known as differential equations with a retarded argument, or DARs for short. When the unknown function at the delayed argument takes a derivative, we have a neutral delay differential equation (NDDE).*

Keywords - Deviating Arguments, Differential Equations, Mathematical equations, delay differential equations, neutral delay differential equation

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INTRODUCTION

Derivatives and integrals are used in mathematical equations to describe the most fascinating natural phenomenon. If the equations are integral or differential, they are referred to as such. There are many distinct types of differential equations that arise in numerous fields, for as in the fields of biology and engineering. When we find a differential equation while analyzing a given system, it is referred to as modeling the system using differential equations. Ordinary differential equations are a common technique in modeling. ODE models presume that systems are independent of their previous states and that their future states are only dependent on the current state. In ODE models, the assumption is made that the system and its subsystems interact instantly and without any delay. Realistic models, on the other hand, incorporate a small bit of lag. As a result, the past, present, and sometimes the derivative of the past states must be taken into account while determining the future state. The functional differential equations (FDEs) represent these models (FDEs). Many models are better characterized by FDEs rather than ODEs because we presume that the system's history has an impact on its future state. The simplest form of FDEs are known as delay differential equations (DDEs). Because the unknown function's derivative depends on the past, they are known as differential equations with retarded arguments. It is natural to extend the DDEs to include NDDEs, in which the unknown function is derived at the delayed argument.

Some special types of differential equations have intriguing features, and one of these types is differential equations with piecewise constant or continuous arguments. (EPCA). There have been several studies on EPCA, which are a hybrid of difference equations and differential equations, that is, a system that can have both discrete and continuous delays. [40]

APPLICATION OF THE METHODS FOR AN ORDINARY DIFFERENTIAL EQUATION (ODE)

Using Newton's second law of motion, an initial value problem for a second order ordinary differential equation can be stated for a basic problem of identifying the location of a damped harmonic oscillator in relation to time under the influence of an external force. Quantitative description of the world around us:

t −→ time variable, t ≥ 0.

x(t) −→ position or displacement of the oscillator, a bounded

and infinitely differentiable function of t.

x'(t) −→ velocity of the oscillator.

X''(t) −→ acceleration of the oscillator.

m −→ mass of the oscillator.

−2mx −→ restoring force acting on the oscillator.

−2mx'(t) −→ damped force acting on the oscillator.

me−t −→ external force acting on the oscillator.

x(0) = 1 −→ initial displacement.

x0(0) = 0 −→ initial velocity.

Then the initial value problem is

$$\begin{aligned} mx''(t) &= -2mx'(t) - 2mx(t) + me^{-t}, \quad t \geq 0 \\ & x(0) = 1,\ x'(0) = 0 \\ \text{or} \quad x''(t) &= -2x'(t) - 2x(t) + e^{-t}, \quad t \geq 0 \\ & x(0) = 1,\ x'(0) = 0. \end{aligned}$$

By Picard's theorem, the initial value issue has a single solution because the coefficients of x'(t), x(t), and et are all continuous functions. First, the problem can be reduced to a simpler problem without the damping term as follows::

Put $y(t) = e^t x(t)$ in

$$\begin{aligned} & x''(t) + 2x'(t) + 2x(t) = e^{-t}, \qquad x(0) = 1,\quad x'(0) = 0 \\ \Leftrightarrow\ & \frac{d^2y}{dt^2} + y = 1,\quad y(0) = 1, \qquad y'(0) = 1. \end{aligned}$$

Application of Adomian Decomposition Method to ODE

Let us apply Adomian decomposition series [1]-[3] $y(t) = 1 + \sum_{n=1}^{\infty} y_n(t)$ on both sides of $y''(t) + y(t) = 1$ with the initial condition $y(0) = 1 \text{ and } y'(0) = 1,$ to get

$$\sum_{n=1}^{\infty} \frac{d^2y_n}{dt^2} + \sum_{n=1}^{\infty} y_n(t) = 0 \quad \text{or} \quad \frac{d^2y_1}{dt^2} + \sum_{n=2}^{\infty} \frac{d^2y_n}{dt^2} = 0 - \sum_{n=1}^{\infty} y_n(t).$$

A simple iteration,

$$\begin{aligned} \frac{d^2y_1}{dt^2} &= 0, \quad y_1(0) = 0,\ y_1'(0) = 1. \\ \frac{d^2y_k}{dt^2} &= -y_{k-1}, \quad y_k(0) = 0 = y_k'(0), \quad k = 2, 3, 4, \dots, \end{aligned}$$

will readily yield

$$y_1 = t,\ y_2 = -\frac{t^3}{3!},\ \dots,\ y_k = (-1)^{k-1}\frac{t^{2k-1}}{(2k-1)!},\ \dots.$$

Hence $y(t) = 1 + t - \frac{t^3}{3!} + \cdots + (-1)^{k-1}\frac{t^{2k-1}}{(2k-1)!} + \cdots = 1 + \sin t$

and $x(t) = e^{-t}y(t) = e^{-t}(1 + \sin t)$ is the desired exact solution.

Application of Laplace Transform Method to ODE

Let us apply Laplace transform on both sides of

$$y''(t) + y(t) = 1,$$

with the initial condition $y(0) = 1$ and $y'(0) = 1$, to get

$$\begin{aligned} s^2L\{y(t)\} - s - 1 + L\{y(t)\} &= \frac{1}{s} \\ L\{y(t)\} &= \frac{s+1}{s^2+1} + \frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{1}{s^2+1} \\ y(t) &= 1 + \sin t. \end{aligned}$$

Hence we get the desired exact solution

$$x(t) = e^{-t}y(t) = e^{-t}(1 + \sin t).$$

Application of Laplace Decomposition Method to ODE

Let us apply decomposition series technique in the Laplace transform method, we take straightaway the equation

$$L\{y(t)\} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^2}L\{y(t)\}.$$

The Laplace decomposition series [5] can be taken as

$$\begin{aligned} L\{y(t)\} &= \frac{1}{s} + \sum_{n=1}^{\infty} L\{y_n(t)\} \\ \frac{1}{s} + \sum_{n=1}^{\infty} L\{y_n(t)\} &= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^2}\left[\frac{1}{s} + \sum_{n=1}^{\infty} L\{y_n(t)\}\right] \\ L\{y_1(t)\} + \sum_{n=2}^{\infty} L\{y_n(t)\} &= \frac{1}{s^2} - \frac{1}{s^2}\sum_{n=2}^{\infty} L\{y_{n-1}(t)\}. \end{aligned}$$

We can obtain an iteration to compute L {yn(t)} as follows :

$$L\{y_1(t)\} = \frac{1}{s^2}$$
$$L\{y_2(t)\} = -\frac{1}{s^2}L\{y_1(t)\} = -\frac{1}{s^4}$$
$$\vdots$$
$$L\{y_n(t)\} = -\frac{1}{s^2}L\{y_{n-1}(t)\} = \frac{(-1)^{n-1}}{s^{2n}}$$
$$\vdots$$

Hence by using Laplace decomposition series for L {y(t)}, we arrive at

$$L\{y(t)\} = \frac{1}{s} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{s^{2n}}, \quad s > 1$$
$$= \frac{1}{s} + \frac{1}{s^2+1}$$
$$y(t) = L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\} = 1 + \sin t$$

and hence the desired solution is

$$x(t) = e^{-t}y(t) = e^{-t}(1+\sin t)$$.

EXISTENCE AND UNIQUENESS OF SOLUTIONS

Differential equations with diverging arguments provide the finest description of realistic models. There are two fundamental issues in studying mathematical models: the existence of solutions and their uniqueness. Deviating arguments and unique solutions are two of the main topics covered in this chapter, so we'll focus on those in this section. Illustrations include the following examples.

We consider the following NDDE:

$$x'(t) = f(t, x(t), x([t]), x'([t])) \quad (3.1)$$
$$x(0) = x_0, \quad (3.2)$$

where $f \in \mathcal{C}(J \times \mathbb{R}^3, \mathbb{R})$, $J = [0, T]$, and [.] is the greatest integer function. Let $\mathcal{D}$ denotes the class of all functions $x : J \to \mathbb{R}$, satisfying

(1) $x(t)$ is continuous, $\forall\, t \in J$.
(2) $x'(t)$ exists and is continuous on the intervals $[n, n+1)$, for $n = 0, 1, 2, \ldots, \tilde{T} - 2$ and on $[\tilde{T} - 1, T)$,

where,

$$\tilde{T} = \begin{cases} [T]+1, & T \neq [T], \\ T, & T = [T]. \end{cases}$$

A function $x : J \to \mathbb{R}$ is said to be a solution of (3.1) if $x \in \mathcal{D}$ and satisfies (3.1) and (3.2) with $x'(t) = x'_+(t)$, the right-hand derivative on $t = 1, 2, \ldots, \tilde{T} - 1$.

A local existence and uniqueness theorem for (3.1) is established in this section using the method of successive approximations. Later, the Lipschtiz type condition on f is relaxed, and the existence of a solution is proven.

LEMMA 1.x(t) is a solution of (3.1), (3.2) on J if and only if x(t)is a solution of

$$x(t) = x_0 + \int_0^t f(s, x(s), x([s]), x'([s]))\,ds. \quad (3.3)$$

PROOF. If $x(t)$ is a solution of (3.1), (3.2) then it follows that $x(t)$ satisfies (3.3). Let $x(t)$ satisfy (3.3). Then at $t = 0$, $x(0) = x_0$ and $f(t, x(t), x([t]), x'([t]))$ is continuous in $(t, x(t), x([t]), x'([t]))$. Differentiating both sides of (3.3) it is seen that $x'(t) = f(t, x(t), x([t]), x'([t]))$. This completes the proof. ∎

Note: In what follows ||.|| means Eucledian norm unless otherwise specified.

THEOREM 3.2.2. Suppose that f satisfies the following:

(A1) $f(t,x,y,z)$ be piecewise continuous function on
$D = \{0 \le t \le T, ||x - x_0|| \le b, ||y - x_0|| \le b, ||z - z_0|| \le c;\ b, c, T > 0\} \subset \mathbb{R}^4$,
(A2) f is bounded on D i.e. $||f|| = sup_{(t,x,y,z)\in D}|f| \le M$; where $M > 0$ is some constant.
(A3) For $x_1 \ge x_2$, $y_1 \ge y_2$, $z_1 \ge z_2 \in D$, and L_1, L_2 positive constants and $0 \le L_3 < 1$. $f(t,x,y,z)$ satisfy the condition,

$$||f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)|| \le L_1||x_1 - x_2|| + L_2||y_1 - y_2|| + L_3||z_1 - z_2||.$$

Then, for $0 \le \beta \le T$ such that $0 \le t \le \beta$ there exists a unique solution $x(t)$ to the IVP (3.1), (3.2).

PROOF. Let $x(t) \in \mathcal{D}$. We define $||x(t)|| = sup_{t\in J}|x(t)|$, where $J = [0, T]$. Then $(\mathcal{C}(J, \mathbb{R}), ||.||)$ is a Banach space. Choose β, ρ, and $\gamma \ge 1$ such that $0 \le \beta \le T$, $0 \le \rho \le b$, $0 \le \gamma \le c$ and $\beta M < \alpha$ where $\alpha = min\{\beta, \frac{\delta}{M}\}$, $\delta = min\{\rho, \gamma\}$.
Let $D_1 = \{x \in \mathcal{C}([0, \beta], \mathbb{R}), ||x(t) - x_0|| \le \alpha\}$, and $x(0) = x_0$ be any element of D_1.
D_1 is a closed, convex, bounded subset of the Banach space $\mathcal{C}(J, \mathbb{R})$.
We now define the map $P : D_1 \to D$ with

$$x_k(t) = (Px_{k-1})(t) = x_0 + \int_0^t f(s, x_{k-1}(s), x_{k-1}([s]), (x_{k-1})'([s]))\,ds \quad (3.4)$$

$x_k([t]) = (Px_{k-1})([t]); k = 1, 2, 3, \ldots$.
By using Leibnitz-Newton theorem for $t \in [n, n+l]$, where $0 < l < 1$ and $n = 0, 1, 2, \ldots, \tilde{\beta} - 2$, the map

$$x_k^n(t) = (Px_{k-1}^n)(t) = x_{k-1}^n(n) + \int_n^t f(s, x_{k-1}^n(s), x_{k-1}^n(n), (x_{k-1}^n)'(n))\,ds. \quad (3.5)$$

$x_k^n(n) = (Px_{k-1}^n)(n); k = 1, 2, 3, \ldots$ is continuously differentiable on $[n, n+l]$ and where,

$$\tilde{\beta} = \begin{cases} [\beta]+1, & \beta \neq [\beta], \\ \beta, & \beta = [\beta]. \end{cases}$$

We shall establish that x_k converges to fixed point on $[0, \beta]$. In view of the continuity of $f(t, x_{k-1}(t), x_{k-1}([t]), (x_{k-1})'([t]))$ on $[0, \beta]$, where $k = 1, 2, \ldots$, it follows that the functions $x_0, x_1(t), \ldots, x_k(t)$ are well defined and are continuous on $[0, \beta]$. It is obvious that, $x_0 \in D_1$. We now show that $x_k(t) \in D_1$. For $t \in [0, \beta]$, we have

$$||x_1(t) - x_0|| \le Mt \le M\beta < \alpha, \quad (3.6)$$

which implies that $x_1(t) \in D_1$. Suppose $x_{k-1}(t) \in D_1$, then

$$x_k(t) = x_0 + \int_0^t f(s, x_{k-1}(s), x_{k-1}([s]), (x_{k-1})'([s]))\,ds.$$

Now $||x_k(t) - x_0|| \le M\beta < \alpha$ and therefore $x_k(t) \in D_1$, $k = 1, 2, 3, \ldots$.
To establish the convergence of the sequence of functions $\{x_k(t)\}$, we take the difference between the successive approximations. For $t \in [0, \beta]$, set

$$p_k(t) = x_k(t) - x_{k-1}(t).$$

Then we have, by (3.6) $||p_1(t)|| \leq Mt$. Since f satisfies (A3) on D_1, it follows,

$$\begin{aligned}||p_2(t)|| &= ||x_2(t) - x_1(t)|| \\ &= ||(Px_1)(t) - (Px_0)(t)||\end{aligned}$$

$$\begin{aligned}
&= ||\int_0^t [f(s,x_1(s),x_1([s]),x_1'([s]))ds] \\
&\quad - \int_0^t [f(s,x_0(s),x_0([s]),x_0'([s]))]ds|| \\
&\leq \int_0^t ||[f(s,x_1(s),x_1([s]),x_1'([s])) - [f(s,x_0(s),x_0([s]),x_0'([s]))]||ds \\
&\leq \int_0^t [L_1||x_1(s) - x_0(s)|| + L_2||x_1([s]) - x_0([s])||]ds \\
&\quad + \int_0^t L_3||x_1'([s]) - x_0'([s])||ds \\
&\leq \int_0^t 2L||x_1(s) - x_0(s)||ds + \int_0^t L\epsilon_1 e^{-\delta}||x_1'([s]) - x_0'([s])||ds \\
&\leq \frac{2LMt^2}{2} + L\epsilon_1 e^{-\delta}Mt \\
&\leq \frac{2LM(t+1)^2}{2} + L\epsilon_1 e^{-\delta}M(t+1)^2 \\
&\leq \frac{M(t+1)^2}{2!}[2L + 2L\epsilon_1 e^{-\delta}] \\
&\leq \frac{2LM(t+1)^2}{2!}[1 + \epsilon_1 e^{-\delta}] \\
&\leq \frac{4LM(t+1)^2}{2!}.
\end{aligned}$$

where $L = max\{L_1, L_2\}$, $L_3 = L\epsilon_1 e^{-\delta}$, where ϵ_1 is sufficiently small.
Similarly,
$||p_3(t)|| \leq \frac{\{4L)\}^2 M(t+1)^3}{3!}$.

A simple induction argument shows that, in general, for $t \in [0,\beta]$

$$||p_k(t)|| \leq \frac{\{4L\}^{k-1}M(t+1)^k}{k!} \leq \frac{\{4L\}^{k-1}M(\beta+1)^k}{k!}. \quad (3.7)$$

Now, consider an infinite series of the form

$$x(t) = x_0 + \sum_{i=1}^{\infty} p_i(t). \quad (3.8)$$

The k-th partial sum of this series is $x_k(t)$, i.e.

$$x_k(t) = x_0 + \sum_{i=1}^{k} p_i(t). \quad (3.9)$$

Therefore, the sequence $\{x_k(t)\}$ converges iff (3.8) converges.
From inequality (3.7), we have

$$x_0 + \sum_{i=1}^{\infty} ||p_i(t)|| \leq x_0 + M\sum_{i=1}^{\infty} \frac{\{4L\}^{i-1}(\beta+1)^i}{i!}. \quad (3.10)$$

Series (3.8) likewise converges (uniformly) on [0, β] according to the comparison test, which shows that the series to the right of (3.10) also converges. Let x be the sum of the (3.8) series (t).

CONTINUOUS DEPENDENCE OF SOLUTIONS

It is shown in this section that the solution x(t) of (3.1) is continually dependent on the beginning conditions under which we get sufficient conditions. We also consider the following equation in addition to (3.1):

$$y'(t) = f(t,y(t),y([t]),y'([t])) + g(t,y(t),y([t]),y'([t])), \quad (3.12)$$

where $f,g \in C(D,\mathbb{R})$ where D is same as defined in (A1).

THEOREM 3.3.1. *Consider (3.12) where $f(t,y(t),y([t]),y'([t]))$ satisfies the assumptions of Theorem 3.2.2 and $g(t,y(t),y([t]),y'([t]))$ is an integrable function of t for each fixed y. Suppose $x(t)$ be unique solution of (3.1) on $[0,\beta]$ with initial condition $x(0) = x_0$ for $t = 0$. Then, (3.12) has a unique solution $y(t)$ on $[0,\beta^*]$ with initial condition $y(0) = y_0$ for $t = 0$. Moreover, if $\beta_1 = min\{\beta,\beta^*\}$, then for $\epsilon > 0$, there exists a $\delta(\epsilon,g) > 0$ such that $||x_0 - y_0|| < \delta$ and $||g(t,y(t),y([t]),y'([t]))|| < \delta$ implies*

$$||x(t) - y(t)|| < \epsilon, \ t \in [0,\beta_1].$$

PROOF. Consider

$$\begin{aligned}
||x(t) - y(t)|| &\leq \int_0^t ||x'(t) - y'(t)||, \\
&\leq \int_0^t ||f(s,x(s),x([s]),x'([s])) - f(s,y(s),y([s]),y'([s]))||ds \\
&\quad + \int_0^t ||g(s,y(s),y([s]),y'([s]))||ds, \\
&\leq 2L\int_0^t ||x(s) - y(s)||ds + L\epsilon_1 e^{-\delta}\int_0^t ||x'(s) - y'(s)||ds \\
&\quad + \int_0^t ||g(s,y(s),y([s]),y'([s]))||ds, \\
&\leq \delta\beta_1 + L\epsilon\epsilon_1 e^{-\delta} + 2L\int_0^t ||x(s) - y(s)||ds.
\end{aligned}$$

Applying Gronwall's inequality, it follows that

$$||x(t) - y(t)|| \leq (\delta\beta_1 + L\epsilon\epsilon_1 e^{-\delta})e^{2Lt}.$$

Choosing suitable δ, it is easy to see that ||x(t) − y(t)|| < e for t∈ [0, β1]. This completes the proof of the theorem.

THEOREM 3.3.2. Consider (3.1) where f(t,y(t),y([t]),y 0 ([t])) satisfies the assumptions of Theorem 3.2.2. Let x(t) be solution of (3.1) on [0, β] with initial condition x(0) = x0 for t = 0 and y(t) be solution of (3.1) on [0, β ∗] with initial condition y(0) = y0 for t = 0. Moreover, if β1 = min{β, β ∗}, then for e > 0, there exists a δ(e) > 0 such that ||x0 − y0|| < δ implies

$$||x(t) - y(t)|| < \epsilon, \ t \in [0,\beta_1].$$

The proof is similar to Theorem 3.3.1 with function g = 0

CONCLUSION

Mathematical equations use derivatives and integrals to describe some of nature's most amazing phenomena. These equations can be categorized as either differential or integral equations, depending on the case. Differential equations, both linear and nonlinear, appear often in the physical, biological, social, and engineering sciences. When we discover a differential equation when researching a system, we are referring to the system's differential equation modeling. Odd-order delay differential equations have been given more attention than even-order ones in the study of oscillatory behavior, contrary to popular opinion. This study focuses on odd-order equations with divergent arguments. More than one criterion is provided for us to verify the oscillation's existence.

If you're trying to deal with FDEs, you have to consider the history. First research into FDEs was done during Euler's investigation on the general shape of curves that are identical to their own evolutes. FDEs were studied in the eighteenth and nineteenth century by Bernoulli, Laplace, Poisson, and Condorcet. You should know all of these names. Schmidt studied differential-difference equations in 1911, when they were first introduced. He made a significant contribution to the study of interactions between species by studying changes and oscillations in animal populations in 1928, and later wrote a book about the importance of history in 1931. For the first time in 1977, a comprehensive theory of differential equations with trailing arguments that are piecewise continuos or constant was proposed by A. D Mykhus, arguing that a comprehensive theory was needed. The first piecewise constant argument mathematical model was devised by Busenberg and Cooke in 1982 when they were investigating the biomedical problem of vertically transmitted illnesses. Differential equations with piecewise constant arguments have grown tremendously since then, with various researchers and references referenced]. Neuronal networks, lossless transmission, nuclear reactors and population dynamics are just a few of the fields in which it is widely employed.

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