Journal of Advances and Scholarly Researches in Allied Education
Vol. 17, Issue No. 2, October-2020, ISSN 2230-7540

A Study of Involution of Prime Rings with Commutativity and Left Centralizers

Pramila Mishra[1]*, Dr. Birendra Kumar Chauhan[2]

[1] Research Scholar, Shri Krishna University, Chhatarpur M.P.

[2] Associate professor, Shri Krishna University, Chhatarpur M.P.

Abstract - In the present study we give a brief exposition of some important terminology in the theory of rings and algebras. Examples and counter examples are also included in this study to make the matter presented in the study self-explanatory and to give a clear sketch of the various notions.In the early stages of general ring theory, striking success of that theory were theorems which asserted the commutativity of the ring when the elements of a ring were subjected to certain types of algebraic conditions.A good cross-section of such results, and the techniques needed to obtain them, can be found where further references can be found. Later as the theory evolved, many authors investigated the relationship between the commutativity of the ring R and certain special types of maps on R. In this direction the concept of centralizing and commuting maps is of great importance. A mapping f of R into itself is called centralizing if [f(x), x] ∈ Z(R) holds for all x ∈ R; in the special case when [f(x), x] = 0 holds for all x ∈ R, the mapping f is said to be commuting.

Keywords - Prime Rings, Commutativity, Left Centralizers, algebraic conditions

- X -

INTRODUCTION

In the early stages of general ring theory, striking success of that theory were theorems which asserted the commutativity of the ring when the elements of a ring were subjected to certain types of algebraic conditions. Much of the initial thrust of the work in this area was either authored by Herstein or inspired by his work. The other significant contributors in this direction have been Ashraf, Bell, Hirano, Kezlan, Komatsu, Tominga, Yaqub with a variety of coauthors. A good cross-section of such results, and the techniques needed to obtain them, can be found where further references can be found. Later as the theory evolved, many authors investigated the relationship between the commutativity of the ring R and certain special types of maps on R. In this direction the concept of centralizing and commuting maps is of great importance. A mapping f of R into itself is called centralizing if [f(x), x] ∈ Z(R) holds for all x ∈ R; in the special case when [f(x), x] = 0 holds for all x ∈ R, the mapping f is said to be commuting. The first result in this direction is due to Divinsky, who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphisms. Two years later, Posner established that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, several authors have subsequently refinedand extended these results in various directions and have established the relationship between the commutativity of a ring R and the existence of certain specific additive maps like derivations, centralizers, generalized derivations and automorphisms of R where further references can be found). In Lee and Lee considered Posner's result mentioned above when the ring R is equipped with involution (for symmetric or skew symmetric elements) and consequently provided counter examples that one cannot expect to conclude the commutativity of R even if R is assumed to be a division ring.

On ∗-commuting and ∗-centralizing mappings in rings with involution

With a view to make our text self-contained, we begin with the following definition.

Definition 2.2.2. *Let R be a ring with involution $*$ and S be a nonempty subset of R. A mapping $f : R \to R$ is said to be $*$-centralizing on S if $[f(x), x^*] \in Z(R)$ for all $x \in S$. As a special case, if $[f(x), x^*] = 0$ for all $x \in S$, then f is said to be $*$-commuting on S.*

Definition 2.2.3. *Let R be a ring with involution $*$ and S be a nonempty subset of R. A mapping $f : R \to R$ is said to be skew $*$-centralizing on S if $f(x) \circ x^* \in Z(R)$ for all $x \in S$. As a special case, if $f(x) \circ x^* = 0$ for all $x \in S$, then f is said to be skew $*$-commuting on S.*

Notice that for any central element a, the map x 7→ ax∗ is ∗-commuting and ∗- centralizing but neither commuting nor centralizing on R. Thus, it is reasonable to study the behaviour of such mappings in the setting of prime and semiprime rings with involution. Over the last three decades, several authors have investigated the relationship between the commutativity of a ring and the existence of certain specific types of maps on R. The first result in this direction is due to Divinsky, who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphisms. This result was subsequently refined and extended by a number of authors in various directions, where further references can be looked). In the year 1957, Posner proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative.

The result which we want to refer states as follows:

Theorem 2.2.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $[d(x), x^*] \in Z(R)$ for all $x \in R$ and $d(S(R) \cap Z(R)) \neq (0)$, then R is commutative.*

In order to develop the proof of the above theorem, we begin with following:

Lemma 2.2.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If $S(R) \cap Z(R) \neq (0)$ and R is normal, then R is commutative.*

Proof. By the hypothesis, we have R is normal, that is $hk = kh$ *for all* $h \in H(R)$ *and* $k \in S(R)$. Let x be an arbitrary element of R. Then $x - x^* \in S(R)$, hence

$$h(x - x^*) = (x - x^*)h \text{ for all } x \in R \text{ and } h \in H(R). \qquad (2.2.1)$$

Also for $s \in S(R) \cap Z(R)$, $s(x + x^*) \in S(R)$ for all $x \in R$. This gives $hs(x + x^*) = s(x + x^*)h$ for all $x \in R$ and $h \in H(R)$. This can be further written as $s(h(x + x^*) - (x + x^*)h) = 0$ for all $x \in R$, $h \in H(R)$. Since the centre of a prime ring is free from zero divisors, we have either $s = 0$ or $h(x + x^*) - (x + x^*)h = 0$ for all $x \in R$ and $h \in H(R)$. Since $S(R) \cap Z(R) \neq (0)$, from the last expression we conclude that

$$h(x + x^*) = (x + x^*)h \text{ for all } x \in R \text{ and } h \in H(R). \qquad (2.2.2)$$

Adding (2.2.1) and (2.2.2), we get $2hx = 2xh$ for all $x \in R$ and $h \in H(R)$. Since $char(R) \neq 2$, we obtain $hx = xh$ *for all* $x \in R$ *and* $h \in H(R)$. Since $x + x^* \in H(R)$, the last relation yields that

$$(x + x^*)y = y(x + x^*) \text{ for all } x, y \in R. \qquad (2.2.3)$$

Again since for $s \in S(R) \cap Z(R)$, $s(x - x^*) \in H(R)$, we conclude that $s((x - x^*)y - y(x - x^*)) = 0$ *for all* $x, y \in R$. Using the fact that the centre of a prime ring is free from zero divisors, we get $s = 0$ or $(x - x^*)y = y(x - x^*)$ for all $x, y \in R$. But $S(R) \cap Z(R) \neq (0)$, so we are forced to conclude that

$$(x - x^*)y = y(x - x^*) \text{ for all } x, y \in R. \qquad (2.2.4)$$

Adding (2.2.3) and (2.2.4), we obtain $2xy = 2yx$. Since $char(R) \neq 2$, the last expression yields that $xy = yx$ for all $x, y \in R$. This proves R is commutative. □

ON DERIVATIONS IN PRIME RINGS WITH INVOLUTION

In [76], Herstein proved that a prime ring R of characteristic not two with a nonzero derivation d satisfying d(x)d(y) = d(y)d(x) for all x, y ∈ R, must be commutative. Further, Daif [54] showed that if a 2-torsion free semiprime ring R admits a derivation d such that d(x)d(y) = d(y)d(x) for all x, y ∈ I, where I is a nonzero ideal of R and d is nonzero on I, then R contains a nonzero central ideal. Further this result was extended by many authors (viz.; [7, 21, 78], where further refrences can be found). In view of Herstein's result [76, Theorem 2] mentioned above, it is a natural question: What can we say about the commutativity of a prime ring if we replace y by x ∗ in the above mentioned condition ? In this direction, we have succeeded in establishing the following result:

Theorem 2.3.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $[d(x), d(x^*)] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. By the assumption, we have

$$[d(x), d(x^*)] = 0 \text{ for all } x \in R. \qquad (2.3.1)$$

A lineralization of (2.3.1) yields that

$$[d(x), d(y^*)] + [d(y), d(x^*)] = 0 \text{ for all } x, y \in R. \qquad (2.3.2)$$

Replacing y by xx^* in (2.3.2), we arrive at

$$\begin{aligned} 0 &= [d(x), d(x)x^* + xd(x^*)] + [d(x)x^* + xd(x^*), d(x^*)] \\ &= d(x)[d(x), x^*] + [d(x), x]d(x^*) + x[d(x), d(x^*)] \\ &+ [d(x), d(x^*)]x^* + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) \\ &= d(x)[d(x), x^*] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) \end{aligned}$$

for all $x \in R$. That is,

$$d(x)[d(x), x^*] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) = 0 \qquad (2.3.3)$$

for all $x \in R$. Replacing x by $x + h'$, where $h' \in H(R) \cap Z(R)$, we obtain

$$d(h')[d(x), x^*] + [d(x), x]d(h') + d(h')[x^*, d(x^*)] + [x, d(x^*)]d(h') = 0.$$

This can be further written as

$$d(h')([d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)]) = 0$$

for all $h' \in H(R) \cap Z(R)$ and $x \in R$. Since the centre of a prime ring is free from zero divisors we get either $d(h') = 0$ for all $h' \in H(R) \cap Z(R)$ or $[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0$ for all $x \in R$. Suppose

$$d(h') = 0 \text{ for all } h' \in H(R) \cap Z(R). \qquad (2.3.4)$$

Replacing h' by $(k')^2$ in (2.3.4), where $k' \in S(R) \cap Z(R)$, we get

$$0 = d(h') = d((k')^2) = d(k')k' + k'd(k') = 2d(k')k'.$$

Since $char(R) \neq 2$, we arrive at

$$d(k')k' = 0 \text{ for all } k' \in S(R) \cap Z(R).$$

Now since the centre of a prime ring is free from zero divisors, we get for each $k' \in S(R) \cap Z(R)$ either $d(k') = 0$ or $k' = 0$. Since $k' = 0$ implies $d(k') = 0$, we may write

$$d(k') = 0 \text{ for all } k' \in S(R) \cap Z(R). \qquad (2.3.5)$$

Let $x \in Z(R)$. Since $char(R) \neq 2$, every $x \in Z(R)$ can be represented as $2x = h + k$, where $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. This implies that $2d(x) = d(2x) = d(h + k) = d(h) + d(k) = 0$. Since $char(R) \neq 2$, we get

$$d(x) = 0 \text{ for all } x \in Z(R). \qquad (2.3.6)$$

Replacing y by $k'y$ in (2.3.2), where $k' \in S(R) \cap Z(R)$ and using (2.3.6), we arrive at

$$k'(-[d(x), d(y^*)] + [d(y), d(x^*)]) = 0 \text{ for all } k' \in S(R) \cap Z(R) \text{ and } x, y \in R.$$

Using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$-[d(x), d(y^*)] + [d(y), d(x^*)] = 0 \text{ for all } x, y \in R. \qquad (2.3.7)$$

On comparing (2.3.2) and (2.3.7), we obtain $2[d(x), d(y^*)] = 0$. Replacing y by y^* and using the fact that $char(R) \neq 2$, we conclude that $[d(x), d(y)] = 0$ for all $x, y \in R$. Therefore in view of [76], we get R is commutative. Now we consider the case

$$[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0 \text{ for all } x \in R.$$

Replacing x by $h + k$, where $h \in H(R)$ and $k \in S(R)$, we get $4[d(k), h)] = 0$. Since $char(R) \neq 2$, we obtain

$$[d(k), h] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \quad (2.3.8)$$

Replacing h by k_0k', where $k_0 \in S(R)$ and $k' \in S(R) \cap Z(R)$, we arrive at $([d(k), k_0])k' =$

0. Using the primeness of R and since $S(R) \cap Z(R) \neq (0)$, we get

$$[d(k), k_0] = 0 \text{ for all } k, k_0 \in S(R). \quad (2.3.9)$$

Now since $char(R) \neq 2$, every $x \in R$ can be represented as $2x = h+k$, where $h \in H(R)$, $k \in S(R)$, so in view of equations (2.3.8) and (2.3.9), we are forced to conclude that

$$[d(k), x] = 0 \text{ for all } k \in S(R) \text{ and } x \in R. \quad (2.3.10)$$

In the year 1995, Bell and Daif showed that if R is a prime ring admitting a nonzero derivation d such that d([x, y]) = 0 for all x, y ∈ R, then R is commutative. This result was extended for semiprime rings by Daif. Further, for semiprime rings, Andima and Pajoohesh showed that an inner derivation satisfying the above mentioned condition on a nonzro ideal of R must be zero on that ideal. Moreover, for semiprime rings with identity, they generalized this result to inner derivations of powers of x and y. Recently, many authors have obtained commutativity of prime and semiprime rings satisfying certain differential identities. In this section, we study the above mentioned result and some other results in the setting of prime rings with involution.

CONCLUSION

The purpose of the present study, R will represent an associative ring with center Z(R). As usual, the commentator xy − yx will be denoted by [x, y]. Given an integer n ≥ 2, a ring R is said to be n-torsion free if nx = 0 (where x ∈ R) implies that x = 0. A ring R is called prime if aRb = (0) (where a, b ∈ R) implies a = 0 or b = 0 and is called semiprime ring if aRa = (0) (where a ∈ R) implies a = 0. An additive map x 7→ x ∗ of R into itself is called an involution if (i) (xy) ∗ = y ∗x ∗ and (ii) (x ∗)∗ = x hold for all x ∈ R. A ring equipped with an involution is called ring with involution or ∗-ring. An element x in a ring with involution is said to be Hermitian if x ∗ = x and skew-Hermitian if x ∗ = −x. The sets of all Hermitian and skew-Hermitian elements of R will be denoted by H(R) and S(R), respectively. The involution is said to be of the first kind if Z(R) ⊆ H(R), otherwise it is said to be of the second kind. In the latter case, S(R) ∩ Z(R) 6= (0). If R is 2-torsion free then every x ∈ R can be uniquely represented in the form 2x = h + k where h ∈ H(R) and k ∈ S(R). Note that in this case x is normal, i.e., xx∗ = x ∗x, if and only if h and k commute. If all elements in R- are normal, then R is called a normal ring.

REFERENCES

[1] Abu-Khuzam, Bell, H. E. and Yaqub, A., Commutativity of rings satisfying certain polynomial identities, Bull. Austral.Math. Soc. 44(1) (1991), 63-69.

[2] Ali, Shakir, On generalized ∗-derivations in ∗-rings, Palest. J. Math. 1 (2012), 32-37.

[3] Ali, Shakir and Dar, N. A., On∗-centralizing mappings in rings with involution, Georgian Math. J. 21(3) (2014)DOI 10.1515/gmj-2014-0006.

[4] Ali, Shakir and Foˇsner, A., On Jordan (α, β) ∗ -derivations in semi-prime ∗-rings, Int. J. Algebra 4(1-4) (2010), 99-108.

[5] Ali, Shakir and Haetinger, C., Jordan α-centralizers in rings and some applications, Bol. Soc. Parana. Mat. (3) 26(1-2) (2008), 71-80.

[6] Ali, Shakir and Salahuddin Khan, M., On∗-bimultipliers, Generalized ∗- biderivations and related mappings, Kyungpook Math. J. 51(3) (2011), 301-309.

[7] Ali, Shakir and Shuliang, H., On derivations in semiprime rings, Algebr. Represent. Theory 15(6) (2012), 1023–1033.

[8] Ali, Shakir, Dar, N. A. and Vukman, J., Jordan left ∗-centralizers of prime and semiprime rings with involution, Beitr Algebra Geom. 54 (2013) 609-624, DOI 10.1007/s13366-012-0117-3.

[9] Ali, Shakir, Fosner, A., Fosner, M. and Khan, M. S., On generalized Jordan triple (α, β) ∗ -derivations and related mappings, Mediterr. J. Math. 10(4) (2013), 1657-1668.

[10] Ali, A. and Yasen, M., A note on automorphisms of prime and semiprime rings, J. Math. Kyoto Univ. 45(2) (2005), 243-246.

[11] Andima, S. and Pajoohesh, H., Commutativity of rings with derivations, Acta Math.Hungar. 128(1-2) (2010), 1-14.

[12] Ara, P. and Mathieu, M., An application of local multipliers to centralizing mappings of ∗-Algebras, Quart. J. Math. Oxford Ser. (2) 44(174) (1993), 129–138.

[13] Ara, P. and Mathieu, M., Local Multipliers of C∗-Algebras, Springer Monograph in

Mathematics, Springer-Verlag, London, (2003).

[14] Arga¸c, N., On prime and semiprime rings with derivations, Algebra Colloq. 13(3) (2006), 371–380.

Corresponding Author

Pramila Mishra*

Research Scholar, Shri Krishna University, Chhatarpur M.P.