Journal of Advances and Scholarly Researches in Allied Education
Vol. 18, Issue No. 4, July-2021, ISSN 2230-7540

A Study of Applications and their Involution of left ∗-Centralizers and ∗-Mappings in Rings

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Abstract - The commutativity of a ring and the existence of certain specific types of maps on R. The first result in this direction is due to prove that a simple artinian ring is commutative if it has a commuting non-trivial automorphisms. This result was subsequently refined and extended by a number of authors in various directions (see for example, where further references can be looked). In the year 1957, Posner proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Besides proving some other results on prime and semiprime rings with involution, the main objective of this study is to present a ∗-version of Posner's theorem. The purpose of the study is to prove that a prime ring R of characteristic not two with a nonzero derivation d satisfying d(x)d(y) = d(y)d(x) for all x, y ∈ R, must be commutative. Further, Daif showed that if a 2-torsion free semiprime ring R admits a derivation d such that d(x)d(y) = d(y)d(x) for all x, y ∈ I, where I is a nonzero ideal of R and d is nonzero on I, then R contains a nonzero central ideal. Further this result was extended by many authors (viz.; where further references can be found).

Keywords - Left ∗-Centralizers, ∗-Mappings in Rings, semiprime rings

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INTRODUCTION

An additive mapping T : R → R is called a left centralizer in case T(xy) = T(x)y holds for all x, y ∈ R. An additive mapping T : R → R is said to be a reverse left centralizer if T(xy) = T(y)x holds for all x, y ∈ R. The definition of a reverse right centralizer should be self-explanatory. For a semiprime ring R, all left centralizers are of the form T(x) = qx for all x ∈ R, where q is an element of Martindale right ring of quotients Qr(R) of R for details). In case if R has an identity element, then T: R → R is a left centralizer if and only if T is of the form T(x) = ax for all x ∈ R and some fixed element a ∈ R. In case T : R → R is a two-sided centralizer, where R is a semiprime ring with extended centoid C, then there exists an element λ ∈ C such that T(x) = λx for all x ∈ R. An additive mapping T : R → R is called a Jordan left centralizer (resp. Jordan right centralizer) if T(x 2) = T(x)x (resp. T(x 2) = xT(x)) holds for all x ∈ R. Clearly, every left centralizer (resp. right centralizer) on a ring R is a Jordan left centralizer on R. But the converse of this statement need not be true in general. However, in case of prime ring of characteristic different from two both concepts coincide. Further, Zalar proved this result in the setting of semiprime ring of characteristic different from two. More related results on centralizers in rings and algebras can be looked where further references can be found.

Let R be a ring with involution ∗. Motivated by the definitions of left (resp. right) centralizer and Jordan left (resp. right) centralizer in rings, Ali and Foˇsner introduced the notion of left (resp. right) ∗-centralizer and Jordan left (resp. right) ∗- centralizer as follows: an additive mapping T : R −→ R is said to be a left ∗-centralizer (resp. Jordan left ∗-centralizer) if T(xy) = T(x)y ∗ (resp. T(x 2) = T(x)x ∗) holds for all x, y ∈ R. The definition of right ∗-centralizer (resp. Jordan right ∗-centralizer) should be self-explanatory. An additive mapping T : R −→ R is said to be a reverse left ∗-centralizer if T(xy) = T(y)x ∗ is fulfilled for all x, y ∈ R. Reverse right ∗-centralizer is defined in a similar way. An additive mapping T : R −→ R is called a ∗-centralizer (resp. reverse ∗-centralizer) if T is both a left and a right ∗-centralizer (resp. reverse left and right ∗-centralizer). Note that for some fixed element a ∈ R, the mapping x 7→ ax∗ is a Jordan left ∗-centralizer and x 7→ x ∗a is a Jordan right ∗-centralizer on R. Clearly, every reverse left ∗-centralizer on a ring R is a Jordan left ∗-centralizer. Thus, it is natural to question that whether the converse of above statement is true. In study is shown that the answer to this question is affirmative if the underlying ∗-ring R is 2-torsion free semiprime ring. Further, we establish a result concerning additive mapping T : R → R satisfying the relation T(x m+n+1) = (x ∗) nT(x)(x ∗) m for all x ∈ R, where m and n are positive integers. Moreover, some nice characterizations of ∗-centralizers in prime and semiprime rings are also given.

On ∗-centralizers in rings

Perhaps, it was Breˇsar and Zalar [49] who first introduced the concept of Jordan centralizers of a ring R, and established that on a prime ring of characteristic different from two every Jordan left centralizer (resp. Jordan right centralizer) is a left centralizer (resp. right centralizer) on R [49, Proposition 2.5]. Further, in [146] Zalar generalized the above mentioned result for semprime ring (without involution). In view of this result, it is natural to question that whether the above result is true in case of ring with involution. In the present study, it is shown that the answer to this question is affirmative if the underlying ∗-ring R is a 2-torsion free semiprime. In fact, we prove the following result:

Proposition 4.2.1. *Let R be a 2-torsion free semiprime ring with involution $*$ and $T : R \to R$ be an additive map which satisfies $T(x^2) = T(x)x^*$ for all $x \in R$. Then, T is a reverse left $*$-centralizer that is, $T(xy) = T(y)x^*$ for all $x,\ y \in R$.*

Proof. By the assumption, we have $T(x^2) = T(x)x^*$ for all $x \in R$. Applying involution $*$ both sides to the above expression, we obtain

$$(T(x^2))^* = x(T(x))^* \text{ for all } x \in R.$$

Define a new map $S : R \to R$ such that $S(x) = (T(x))^*$ for all $x \in R$. Then we see that

$$\begin{aligned} S(x^2) &= (T(x^2))^* \\ &= (T(x)x^*)^* \\ &= x(T(x))^* \\ &= xS(x) \text{ for all } x \in R. \end{aligned}$$

Hence, we obtain S(x 2) = xS(x) for all x ∈ R. Thus, S is a Jordan right centralizer on R. In view S is a right centralizer that is, S(xy) = xS(y) for all x, y ∈ R. This implies that (T(xy))∗ = x(T(y))∗ for all x, y ∈ R. By applying involution to the both sides of the last relation, we find that T(xy) = T(y)x ∗ for all x, y ∈ R. This completes the proof of the proposition.

Theorem 2. Let R be a non-commutative prime ring with involution ∗ such that char$(R) \neq 2$. Then the following conditions are mutually equivalent:

(i) R is normal

(ii) there exists a nonzero commuting Jordan left ∗-centralizer T- on R.

Proof. Suppose R is a normal ring. Then the mapping x 7→ x ∗ is a commuting nonzero Jordan left ∗-centralizer on R. Now suppose (ii) holds, we have to prove R is normal. there exists µ ∈ C and a map v : R → C such that

$$T(x) = \mu x + \nu(x) \text{ for all } x \in R.$$

On the other hand, it follows from the Proposition 4.2.4, T(x) = qx∗ for all x ∈ R, where q ∈Qr(R). Thus, we have

$$qx^* - \mu x \in C \text{ for all } x \in R.$$

Since the identity involves involution, so it is a functional identity or the so-called gidentity. In view of Lemma 1.3.2, we conclude that qx∗ − µx ∈ C for all x ∈ Qs(R), the symmetric ring of quotients. Note that Qs(R) has the identity element 1. Replacing x by 1 in the above expression, we see that q − µ ∈ C. This implies that [q, y] = 0 for all y ∈ Qs(R). Thus,

$$T(x) = \lambda x^* \text{ for all } x \in R,$$

where $\lambda = q \in C$. Since $T \neq 0$, it follows that $\lambda \neq 0$. Thus we conclude that $0 = [T(x), x] = [\lambda x^*, x] = \lambda[x^*, x]$ for all $x \in R$. The primeness of R yields that R is normal. This proves the theorem completely. □

The above theorem has the following interesting consequence:

Corollary. Let R be a non-commutative prime ring with involution ∗ such that char(R) 6= 2. If T is a nonzero Jordan left ∗-centralizer on R such that [T(x), x] = 0for all x ∈ R, then there exists λ ∈ C, the extended centroid of R such that T(x) = λx∗ for all x ∈ R.

Theorem. Let R be a non-commutative prime ring with involution ∗ such that char(R) 6= 2. If T1 and T2 are two nonzero Jordan left ∗-centralizers on R such that T1(x)x − xT2(x) = 0 for all x ∈ R, then R is normal.

Proof. By the given hypothesis, we have

$$T_1(x)x - xT_2(x) = 0 \text{ for all } x \in R. \quad (4.2.1)$$

On linearizing (4.2.1), we get

$$T_1(x)y + T_1(y)x - xT_2(y) - yT_2(x) = 0 \text{ for all } x, y \in R. \quad (4.2.2)$$

Replacing y by yx in (4.2.2) and using Proposition 4.2.1, we arrive at

$$T_1(x)yx + T_1(x)y^*x - xT_2(x)y^* - yxT_2(x) = 0 \text{ for all } x, y \in R. \quad (4.2.3)$$

On Jordan left ∗-centralizers in rings

During the last few decades, there has been ongoing interest concerning the left centralizers (resp. Jordan left centralizers) on prime and semiprime rings. Recently, many authors viz; have obtained some interesting results in rings and algebras. In Vukman provedthat if R is a non-commutative 2-torsion free semiprime ring and S, T: R → R- are left centralizers such that $[S(x), T(x)]S(x) + S(x)[S(x), T(x)] = 0$ for all $x \in R$, In case R is a prime ring and $S \neq 0$ $(T \neq 0)$, then there exists λ ∈ C such that T = λS (S = λT). The intent of this study is to study similar types of problems in the setting of rings with involution by replacing left centralizer with Jordan left ∗-centralizer.

We begin with the following:

Lemma 1. Let R be a non-commutative prime ring with involution ∗ and let T : R −→ R be a Jordan left ∗-centralizer on R. If T(x) ∈ Z(R) for all x ∈ R, then T = 0.

Proof. By the hypothesis, we have [T(x), y] = 0 for all x, y ∈ R. Substituting x 2 for x in the above relation, we obtain

$$\begin{aligned} 0 &= [T(x^2), y] \\ &= [T(x)x^*, y] \\ &= [T(x), y]x^* + T(x)[x^*, y] \text{ for all } x, y \in R. \end{aligned}$$

In view of our hypothesis, the last expression yields that $T(x)[x^*, y] = 0$ for all $x,\ y \in R$. Since the centre of a prime ring is free from zero divisors, either $T(x) = 0$ or $[x^*, y] = 0$. Let $A = \{x \in R|\ T(x) = 0\}$ and $B = \{x \in R|\ [x^*, y] = 0$ for all $y \in R\}$. It can be easily seen that A and B are two additive subgroups of R whose union is R and hence by Brauer's trick, we get $A = R$ or $B = R$. If $B = R$, then R is commutative, which gives a contradiction. Thus the only possibility remains that $A = R$. That is, $T(x) = 0$ for all $x \in R$. This finishes the proof. □

Proposition 4.3.1. *Let R be a noncommutative prime ring with involution ∗ and let $S, T : R \longrightarrow R$ be Jordan left ∗-centralizers. Suppose that $[S(x), T(x)] = 0$ holds for all $x \in R$. If $T \neq 0$, then there exists $\lambda \in C$ such that $S = \lambda T$.*

Proof. By Proposition 4.2.1, we conclude that S and T are reverse left ∗-centralizers on R. In view of the hypothesis, we have

$$[S(x), T(x)] = 0 \text{ for all } x \in R. \qquad (4.3.1)$$

Linearizing (4.3.1), we get

$$[S(x), T(x)] + [S(x), T(y)] + [S(y), T(y)] + [S(y), T(x)] = 0 \qquad (4.3.2)$$

for all $x, y \in R$. Using (4.3.1) in (4.3.2), we have

$$[S(x), T(y)] + [S(y), T(x)] = 0 \text{ for all } x,\ y \in R. \qquad (4.3.3)$$

Replacing x by zx in (4.3.3), we obtain

$$[S(x), T(y)]z^* + S(x)[z^*, T(y)] + [S(y), T(x)]z^* + T(x)[S(y), z^*] = 0 \qquad (4.3.4)$$

for all $x, y, z \in R$. Application of (4.3.3) yields that

$$S(x)[z^*, T(y)] + T(x)[S(y), z^*] = 0 \text{ for all } x, y, z \in R. \qquad (4.3.5)$$

CONCLUSION

In the present study we give a brief exposition of some important terminology in the theory of rings and algebras. Examples and counter examples are also included in this study to make the matter presented in the study self-explanatory and to give a clear sketch of the various notions. The knowledge of some elementary concepts like groups, rings, ideals, fields, modules, homomorphism etcetera have been presumed. Throughout the thesis, unless otherwise mentioned, R will denote an associative ring (may be without unity) containing at least two elements.

Throughout the discussion all rings are associative unless indicated otherwise and Z(R) denotes the center of the ring R. For elements x and y in a ring R, we shall write [x, y] = xy − yx and x ◦ y = xy + yx. The element [x, y] is called the Lie product (or the commutator) of elements x and y, and x ◦ y is called the Jordan product (or the anti-commutator) of x and y. the commutativity of prime rings with involution. An additive mapping x 7→ x ∗ on a ring R is called an involution on R if (xy) ∗ = y ∗x ∗ and (x ∗)∗ = x holds for all x, y ∈ R. A ring equipped with an involution is called a ring with involution or ∗-ring. Let d : R → R be an additive mapping such that d(xy) = d(x)y + xd(y) for all x, y ∈ R. Then d is said to be a derivation on R. Let S be a nonempty subset of R. A mapping f : R → R is called centralizing on S if [f(x), x] ∈ Z(R) for all x ∈ S, and is called commuting on S if [f(x), x] = 0 for all x ∈ S. Motivated by the existence of centralizing mappings in rings, the notions of ∗-centralizing and ∗-commuting mappings in rings with involution.

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