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A Study on Fixed Point Theorem of Contraction Mapping Principle in Metric Spaces

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Abstract - In this study, it is shown that the fixed-point theory is best approximated and the variation inequality results are best approximated. The change of inequality results in a theory of fixed points. It is also shown to be the maximum element in mathematical economics for the fixed-point theory. Ultimately, some earlier results have been proved. We need to discuss the existence of solutions with certain desired properties in many of the problems arising from models of chemical reactors, neutron transport, population biology, infectious diseases, economies and other systems. Banach (1922) was the first mathematician to show that solutions of nonlinear equations existed and existed under certain conditions. Banach's fixed point theorems have become a key feature in functional analysis history. The Banach contraction principle has many applications and has spread over nearly all mathematical branches.In the study of problems of the common fixed points of non-commuting mapping, the notion of compatibility plays an important role. In addition, the continuity of one of the mapping process is compulsorily required when obtaining a common fixed point in the orems. The present study aims to achieve a reciprocal continuity of common fixed-point theorems. Finally, the existence and uniqueness of common solutions in the dynamic programming for the functional equations has been tested.

Keywords - Fixed Point Theorem, Metric Spaces, mathematical economics, nonlinear equations, mapping process

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INTRODUCTION

The present chapter is aimed at obtaining common fixed point theorems using reciprocal continuity. Lastly an attempt has been made to prove the existence and uniqueness of common solutions of the functional equations arising in the dynamic programming. Assume X is normed linear space and A and B are self-mappings on X.

Definition 1.1.1 Two mappings A and B are said to be Rweakly commuting (see Pant [12]) if there exists real number R > 0 such that \\ABx — BAx|| < R\\Ax — Bx\\ for all x G X.

Definition 1.1.2 A and B are said to be compatible if limn-,001|ABxn — BAxn\\ = 0, whenever {xn} is a sequence in X such that limn Axn = limn Bxn = t for some t in X.

Definition 1.1.3 A and B are said to be reciprocally continuous if limn ABxn = At and limn BAxn = Bt whenever {xn} is a sequence in X such that limn Axn = limn Bxn = t for some t in X.

If A and B both are continuous then they are obviously reciprocally continuous but the converse is not true as shown in [9]. It is also clear that H-weakly commuting mappings are compatible,ut the converse is not true as shown in [1]. Let C be convex subset of X. Suppose A, B and T are mappings from C to itself satisfying following conditions:

(H_1) $kA(C) + (1-k)T(C) \subset T(C)$ and

$kB(C) + (1-k)T(C) \subset T(C)$

for some $0 < k < 1$.

(H_2) $||Ax - By|| \leq h \max\{||Tx - Ty||, ||Ax - Tx||, ||By - Ty||, \frac{1}{2}(||Ax - Ty|| + ||By - Tx||)\}$

for all $x, y \in C$ and $0 \leq h < 1$.

Now let xq be arbitary point in C. Since (Hi) holds, we can define a sequence {Txn} by

$$Tx_{2n+1} = kAx_{2n} + (1-k)Tx_{2n} \text{ and}$$

$$Tx_{2n+2} = kBx_{2n+1} + (1-k)Tx_{2n+1} \qquad (1.1)$$

for n = 1,2,3,................

Now we prove the following lemma which is required for main theorems.

Lemma 1.1.1 If $h < \frac{k}{k+1}$ then the sequence {Tzn} defined by (1.1) is a Cauchy sequence.

Proof: Let $d_n = ||Tx_{n+1} - Tx_n||$.

Therefore $d_{2n+1} = k||Bx_{2n+1} - Tx_{2n+1}||$ and

$d_{2n} = k||Ax_{2n} - Tx_{2n}||$. Now

$$\begin{aligned} d_{2n+1} &= ||kBx_{2n+1} + (1-k)Tx_{2n+1} - kAx_{2n} - (1-k)Tx_{2n}|| \\ &\leq k||Bx_{2n+1} - Ax_{2n}|| + (1-k)||Tx_{2n+1} - Tx_{2n}|| \\ &\leq (1-k)d_{2n} + k||Ax_{2n} - Bx_{2n+1}|| \\ &\leq (1-k)d_{2n} + kh\max\{||Tx_{2n} - Tx_{2n+1}||, ||Ax_{2n} - Tx_{2n}||, \\ &\quad ||Bx_{2n+1} - Tx_{2n+1}||, \frac{1}{2}(||Ax_{2n} - Tx_{2n+1}|| \\ &\quad + ||Bx_{2n+1} - Tx_{2n}||)\} \end{aligned}$$

$$\begin{aligned} &\leq (1-k)d_{2n} + kh\max\{d_{2n}, \frac{d_{2n}}{k}, \frac{d_{2n+1}}{k}, \frac{1}{2}(||Ax_{2n} - Tx_{2n}|| \\ &\quad + ||Tx_{2n} - Tx_{2n+1}|| + ||Bx_{2n+1} - Tx_{2n+1}|| \\ &\quad + ||Tx_{2n+1} - Tx_{2n}||)\} \end{aligned}$$

$$\begin{aligned} &\leq (1-k)d_{2n} + kh\max\{d_{2n}, \frac{d_{2n}}{k}, \frac{d_{2n+1}}{k}, \\ &\quad \frac{1}{2}(\frac{d_{2n}}{k} + d_{2n} + \frac{d_{2n+1}}{k} + d_{2n})\} \\ &\leq (1-k)d_{2n} + h\max\left\{d_{2n}, d_{2n+1}, \frac{1}{2}(d_{2n} + d_{2n+1}) + kd_{2n}\right\} \end{aligned}$$

If d_{2n} is maxium then $d_{2n+1} \leq (1-k)d_{2n} + hd_{2n}$ i.e.

$d_{2n+1} \leq (1-k+h)d_{2n}$. Since $h < \frac{k}{k+1}$, $h < k$. Hence $h - k < 0$

i.e. $(1 + h - k) < 1$.

If d_{2n+1} is maximum then $d_{2n+1} \leq (1-k)d_{2n} + hd_{2n+1}$

i.e. $d_{2n+1} \leq \frac{(1-k)}{(1-h)}d_{2n}$. Since $h < \frac{k}{k+1}$, then $h < k$ and $1 - k < 1 - h$

i.e. $\frac{1-k}{1-h} < 1$.

If $\frac{1}{2}(d_{2n} + d_{2n+1}) + kd_{2n}$ is maximum then

$d_{2n+1} \leq (1-k)d_{2n} + \frac{h}{2}(d_{2n} + d_{2n+1}) + hkd_{2n}$ i.e. $d_{2n+1} \leq (\frac{2-2k+h+2kh}{2-h})d_{2n}$.

Since $h < \frac{k}{k+1}$, $-2k + h + 2hk < -h$ i.e. $(\frac{2-2k+h+2kh}{2-h}) < 1$.

Let $\alpha = \max\{(1 + h - k), (\frac{1-k}{1-h}), (\frac{2-2k+h+2hk}{2-h})\}$.

Clearly $0 < \alpha < 1$ and $d_{2n+1} \leq \alpha d_{2n}$.

Similarly we can prove that $d_{2n} \leq \alpha d_{2n-1}$. Now

$$\begin{aligned} d_n = ||Tx_{n+1} - Tx_n|| &\leq \alpha||Tx_n - Tx_{n-1}|| \\ &\leq \alpha^2||Tx_{n-1} - Tx_{n-2}|| \\ &\leq \ldots\ldots \\ &\leq \alpha^n||Tx_1 - Tx_0||. \end{aligned}$$

We have for $n \geq m$,

$$\begin{aligned} ||Tx_m - Tx_n|| &\leq ||Tx_m - Tx_{m+1}|| + ||Tx_{m+2} - Tx_{m+3}|| + \ldots\ldots \\ &\quad \ldots\ldots + ||Tx_{n-1} - Tx_n|| \\ &\leq \alpha^m||Tx_1 - Tx_0|| + \alpha^{m+1}||Tx_1 - Tx_0|| + \ldots\ldots \\ &\quad \ldots\ldots + \alpha^{n-1}||Tx_1 - Tx_0|| \\ &\leq \alpha^m[1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{n-m-1}]||Tx_1 - Tx_0|| \\ &< \frac{\alpha^m}{1-\alpha}||Tx_1 - Tx_0||. \end{aligned}$$

Therefore $||Tx_m - Tx_n|| \to 0$ as $m \to \infty$. Hence $\{Tx_n\}$ is a Cauchy sequence.

COMMON FIXED POINT THEOREMS

Theorem 1.2.1 Let A,B and T be mappings from C to itself satisfying (Hi) and {H-f). If one of the pair (A, T) or (B,T) is reciprocally continuous and compatible, T{C) is complete and $h < \frac{k}{k+1}$ then A, B and T have a unique common fixed point in T{C).

Proof: Let {Txn} be a sequence defined by (1.1) in T{C). By lemma (1.1.1), {'Txn} is Cauchy sequence. Since T{C) is complete {Txn} converges to a point Tz in T{C) i.e. in C. Thus Txn —* Tz. Now as n —>oo, from (1.1) Ax2n —>Tz and Bx2n+1 —» Tz. Suppose (A, T) is reciprocally continuous and compatible, therefore ATxn —* ATz, TAxn -* TTz and |\ATx2n — TAx2n\\ -* 0. Therefore ATz = TTz.

Now using {H2) we have

$$\begin{aligned} ||ATz - BTz|| &\leq h\max\{||TTz - TTz||, ||ATz - TTz||, \\ &\quad ||BTz - TTz||, \frac{1}{2}(||ATz - TTz|| \\ &\quad + ||BTz - TTz||)\} \end{aligned}$$

$$\begin{aligned} &\leq h\max\{0, 0, ||BTz - ATz||, \frac{1}{2}||BTz - ATz||\} \\ &\leq h||ATz - BTz||. \end{aligned}$$

$\therefore ATz = BTz$ since $h < 1$. Hence $ATz = BTz = TTz$.

Again using (H_2) we get

$$\begin{aligned} ||Ax_{2n} - BTz|| &\leq h\max\{||Tx_{2n} - TTz||, ||Ax_{2n} - Tx_{2n}||, \\ &\quad ||BTz - TTz||, \frac{1}{2}(||Ax_{2n} - TTz|| \\ &\quad + ||BTz - Tx_{2n}||)\}. \end{aligned}$$

Letting $n \to \infty$, it follows that

$$\begin{aligned} ||Tz - BTz|| &\leq h \max\{||Tz - TTz||, ||Tz - Tz||, 0, \\ &\qquad \frac{1}{2}(||Tz - TTz|| + ||BTz - Tz||)\} \\ &\leq h||Tz - BTz||. \end{aligned}$$

Since $h < 1$, $Tz = BTz$. Thus $ATz = BTz = TTz = Tz$, i.e. Tz is a common fixed point of A, B and T.

Similarly if (B, T) is reciprocally continuous and compatible then also we can prove that Tz is a common fixed point of A, B and T. To prove uniqueness, if possible suppose that Tz' is other common fixed point of A, B and T. $\therefore \ ATz' = BTz' = TTz' = Tz'$.

Again using (H2) we have

$$\begin{aligned} ||Tz' - Tz|| &= ||ATz' - BTz|| \\ &\leq h \max\{||TTz' - TTz||, ||ATz' - TTz'||, \\ &\qquad ||BTz - TTz||, \frac{1}{2}(||ATz' - TTz|| \\ &\qquad + ||BTz - TTz'||)\} \\ &\leq h \max\{||Tz' - Tz||, 0, 0, \frac{1}{2}(||Tz' - Tz|| \\ &\qquad + ||Tz - Tz'||)\} \\ &\leq h||Tz' - Tz||. \end{aligned}$$

Since $h < 1$, so $Tz' = Tz$. This completes the proof. Some obvious corollaries can be obtained from Theorem 1.2.1 by letting (i) A = B, (ii) A = B and T = /, identity mapping, (iii) k =-l etc. We state few of the corollaries.

Corollary 1.2.1 Let A, *T* be reciprocally continuous and compatible mappings from C to itself satisfying

$$kA(C) + (1-k)T(C) \subset T(C) \text{ and}$$
$$||Ax - Ay|| \leq h \max\{||Tx - Ty||, ||Ax - Tx||, ||Ay - Ty||, \tfrac{1}{2}(||Ax - Ty|| + ||Ay - Tx||)\},$$

where $0 < h, k < 1$ *and* $h < \frac{k}{k+1}$. *If* $T(C)$ *is complete then* A *and* T *have a unique common fixed point in* $T(C)$.

Corollary 1.2.2 Let A, B and T be mappings from C to itself such that A{C) C T{C) and B{C) C T{C) satisfying (H2). If one of the pair (A, T) or (B, T) is reciprocally continuous and compatible, T{C) is complete and $h < \frac{1}{2}$ then A, B and T have a unique common fixed point in T{C).

Theorem 1.2.2 *Let* T_i, T *be mapings from* C *to itself for* $i = 1, 2, 3, \ldots\ldots$ *satisfying*

$$kT_i(C) + (1-k)T(C) \subset T(C), \text{ where } 0 \leq k < 1 \text{ and}$$
$$||T_ix - T_{i+1}y|| \leq h \max\{||Tx - Ty||, ||T_ix - Tx||, ||T_{i+1}y - Ty||, \tfrac{1}{2}(||T_ix - Ty|| + ||T_{i+1}y - Tx||)\} \quad (1.2)$$

where $0 < h < \frac{k}{k+1}$. *Suppose one of the pair* (T_{2i}, T) *or* (T_{2i+1}, T) *is compatible reciprocally continuous and* $T(C)$ *is complete, then* $T, T_1, T_2, \ldots\ldots$ *have a unique common fixed point in* $T(C)$.

Proof: Suppose i is fixed. By Theorem 1.2.1 we conclude that T2i, r2i+i and T have a unique common fixed point in T{C). Now we prove that % and Thave a unique common fixed point in T{C) for i = 1,2,3,...... Suppose z € T{C) is a common fixed point of T,Ti,T2 and w G T(C) is a common fixed point of T,T2,T3 then $Tz = T_1z = T_2z = z$ and $Tw = T_2w = T_3w = w$.

Now using (1.2) it follows that

$$\begin{aligned} ||z - w|| &= ||T_2z - T_3w|| \\ &\leq h \max\{||Tz - Tw||, ||T_2z - Tz||, ||T_3w - Tw||, \\ &\qquad \frac{1}{2}(||T_2z - Tw|| + ||T_3w - Tz||)\} \\ &\leq h \max\left\{||z - w||, 0, 0, \frac{1}{2}(||z - w|| + ||w - z||)\right\} \\ &\leq h||z - w||. \end{aligned}$$

Therefore z = w since h < 1. Thus z is a unique common fixed point of T, Ti, T2, T3. Therefore, by induction, T, Ti, T2,.... have a unique common fixed point in T(C).

METRIC SPACE

METRIC SPACE X be a nonempty set of elements (which we call points) together with a real valued function d defined on X X× such that for all x y z , , in X

$$d(x, y) \geq 0 \quad (1.1)$$

$$d(x, y) = 0 \text{ iff } x = y \quad (1.2)$$

$$d(x, y) = d(y, x) \quad (1.3)$$

$$d(x, y) \leq d(x, z) + d(y, z) \quad (1.4)$$

The function d is called metric and (X, d) is called metric space.

Example. The set R of all numbers with

$$d(x,y) = | x - y | \text{ is a metric space} \qquad (1.5)$$

Definition. Sequence $< x_n >$ of real numbers we mean a function that maps each natural number n into the real number i.e. $f : N \rightarrow R$.

Definition. A sequence $< x_n >$ in a metric space is called a Cauchy sequence if given ε> 0 there exist a positive integer N such that for all $n, m \geq N$

$$d(x_n, x_m) < \varepsilon \qquad (1.6)$$

Definition. A metric space is called complete metric space if every Cauchy sequence in it is convergent.

FUNCTION

If some points of a space X are put in correspondence with some points of another space Y in such a way that one and only one point y Y∊ corresponds to a given point x of a certain subset of X then this correspondence is called a function or mapping. The set of points is called the domain of the function. The set of points of the space Y which corresponds to the points of domain is called the range of the function.

Definition. Let (X, d) denote the complete metric space. The two mappings f and g are mapping from X into itself are said to be commutative or commuting if

$$(fg)(x) = (gf)(x), \quad \forall x \in X,$$

and coincidentally commutating if they commute at coincidence points.

Definition. Two self-mappings f and g are called weak commutating, if

$$f(X) \subset g(X) \text{ and}$$

$$d(f^2g^2x,\ g^2f^2x) \leq d(f^2gx, g^2fx)$$

$$d(fg^2x, g^2fx) \leq d(fgx, gfx)$$

$$\leq d(g^2x, f^2x),\ \forall x \in X$$

Definition. Let ϕ is a mapping from $R^+ \rightarrow R^+$, which satisfies the following conditions:

$$\phi \text{ is increasing}$$

$$\phi(t) < t, \forall t > 0$$

$$\lim_{n\to\infty} \phi^n(t) = 0, \text{ where } \phi^n(t)$$

denotes the composition of $\phi(t)$ with itself n-times.

FIXED POINT THEOREM OR CONTRACTION MAPPING PRINCIPLE

Let f is a mapping from a nonempty set into itself, if $f(x) = x$ then f has a fixed point.

The best well known fixed point theorem is the Banach [6] fixed point theorem, also called the contraction mapping principle which is as follows "every contraction mapping from a complete metric space into itself has a unique fixed point." Banach contraction principle is also stated equivalently.

If (X, d) be a complete metric space. Then by well-known Banach theorem, a mapping $f : X \rightarrow X$ is said to have fixed point if there exists an α∊ (0,1) such that,

$$d(fx, fy) \leq \alpha d(x,y), \quad \forall x, y \in X$$

COUPLED FIXED POINT

An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if

$$x = F(x,y) \text{ and } y = F(y,x)$$

DIFFERENTIAL EQUATION

We can use differential equation in many branches as in the theory of bending beams, oscillation of mechanical system, and condition of heat velocities of chemical reaction. If contains dependent and independent variables and derivative of the dependent variable with respect to the independent variable, there are also different orders of differential equations such as first order, second order up to nth order.

ORDER OF THE EQUATION

The order of the highest derivative in a differential equation is called the order of the differential equation. A differential equation is said to be linear if it has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_n + (x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + a_1 x\frac{dy}{dx} + a_0(x)y = g(x).$$

The equations

$$xdy + ydx = 0$$

$$y'' + 2y' + y = 0$$

are linear first order and second order ordinary differential equation respectively.

CONCLUSION

Fixed point theory is quite useful in the existence theory of differential, integral, partial differential and functional equations. This is a fundamental mathematical tool used to demonstrate the existence of solutions in game and mathematical theory. The best approach theory applications, problems with optimisation, varying inequalities, problems of complementarity and problems of own value are well known. Set point theory is a very important tool in the nonlinear problem field and is a backbone of the nonlinear analysis. It includes various fields such as mathematical economics, game theory, biology, engineering and physics. In the 20th century it laid the scientific foundation of a fixed-point theory. The main result of this theory is the principle of contraction (from the '30s), which generated important research lines and theory applications for functional equations, differential equations, integral equations, etc. Tarki, Bourbaki, Banach, Perov, Luxemburg-Jung, Brwer, Schauder, Tihonov and Brouwder-Ghode-Kirk (Rus, Petruşel, Petruşel 2008) are traditional theorems of this theory. Banach's theory of fixed points, also known as the principle of contraction, is an important tool in metric space theory. It ensures that solutions to form x = f(x) equations exist and are unique for many applications and provides a constructive way of identifying those solutions. for a wide variety of applications.

REFERENCES

[1] Abbas M. and Jungck G.: Common fixed point results for noncommuting mappings without continuity in cone metric space. J. Math. Anal.Appl.; 341, 416-420, (2008).

[2] Abbas M., Rhoades B.E. and NazirTalat: Common fixed points for four maps in cone metric spaces. Applied Mathematics and Computation XXX (2010) xxx-xxx. (Article in Press)

[3] Ahmed Al-Rawashdeh, WasfiShantanawi and MunaKhanaqji: Normed ordered and E-metric spaces I.J. of Math. And Mathematical Sciences.Article ID 272137, 11 pages, (2012).

[4] Azam Akbar, Muhammad Arshad and Ismat Beg: Banach Contraction principle on cone rectangular metric spaces. Appl. Anal. Discrete Math.3, 236-241, (2009).

[5] Bainov D.D. and Simenov P.S.: Impulsive Differential Equations, Periodic Solutions and applications. Pitman Monographs and Surveys in Pure and applied Mathematics. Vol. 66, Pitman, London (1993).

[6] Banach S.: Surless operations dansles ensembles abstraits et leursaux..equations-integrals, Fund. Math.3, 160, (1922).

[7] Basu D .K.: Generalization of fixed point theorem Bull. Cal. Math. Soc. 82, 106-108, (1990).

[8] Border K.C.: Fixed point theorems with application to economics and game theory. Cambridge University Press, Cambridge U.K. (1985).

[9] Boyd D.W. and Wong J.S.W.: On nonlinear contractions Proc. Amer. Math. Soc. 20, 458-464, (1969).

[10] Bridges D., Richman W.H. and Mines R.: Extensions and fixed points of contractive maps in n R . J. Math. Anal.Appl. 165, 438-456, (1992).

[11] Brosowski B.: Fixpunktasatzeinder approximation theorie. Mathematica (Cluj) 11, 195-220, (1969).

[12] Browder F.E. and Petryshyn W.V.: Construction of fixed points of nonlinear mapping in Hilbert Spaces. Press J. Math. Anal.Appl. 20, 197-228, (1967).

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