Solving the Axisymmetric Dirichlet Potential using the One-Variable Hankel Transform of the I-Function

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Abstract - The study presents a novel and efficient approach to solving the axisymmetric Dirichlet potential problem by employing the one-variable Hankel transform of the I-function. The Dirichlet potential problem, a fundamental concept in mathematical physics, arises in various fields including electromagnetism, fluid dynamics, and heat conduction. Traditional solutions often involve complex mathematical manipulations and extensive computational efforts.

In this research, we leverage the powerful tool of the one-variable Hankel transform applied to the I-function, a special function in mathematical analysis. By transforming the governing equations into the Hankel domain, we simplify the problem significantly, reducing it to a manageable form. The transformed equations are solved analytically, leading to explicit solutions for the axisymmetric Dirichlet potential. The efficiency and accuracy of the proposed method are demonstrated through comprehensive numerical simulations and comparisons with existing solutions. Using the Hankel transform of an I-function in one variable, we have solved the famous Axisymmetric Dirichlet problem for a half-space in this study. When dealing with cylindrical coordinates and boundary value issues, the Hankel transform is a powerful tool. We have solved the Axisymmetric Dirichlet problem in a half space defined by the following equations:

Keywords - potential problem, hankel transform, saxena's I- function of one variable, fox's Hfunction of one variable.

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INTRODUCTION

Calculus without limits has attracted the attention of many mathematicians and scientists. In the 1920s of the previous centuries, this kind of calculus first appeared. The book "Quantum Calculus" by Kac and Cheung contains the foundations of q-calculus. Applications in fields such as ordinary and fractional differential equations, finding solutions to q-difference and q-integral equations, and q-transform analysis have boosted the profile of research into q-integrals and q-derivatives of arbitrary order.

From the seventeenth century to the current day, hypergeometric functions developed as a natural unification of many other types of functions that were already known at the time. Basic or q-analogues of these functions may be derived by further generalization utilizing the idea of basic number q. Since its discovery 30 years ago, q-functions have attracted a lot of attention due to their potential applications in number theory, statistics, and other branches of mathematics and science. New generalized versions of fundamental hypergeometric functions have emerged as a result of recent work in their theory. Functions like those of Mac-Roberts, Meijer, Fox, and Saxena, as well as their q-analogues, are included here. In light of the q-extension of the generalized H-function and I-function, the q-gamma function, have developed the q-analogue of the I-function.

To this end, Truesdell has proposed a theory that has produced many conclusions for specific functions that meet the so-called Truesdell's F-equation. resulted in a derivation of the F-equation for a decreasing gradient using this theory. He derived many formulae for the F-equation, including those based on orthogonality, Rodrigue's method, and Schalafli's method. found many variants of the I-function that agree with Truesdell's F-equation in both directions. The curious reader is directed to and the references therein for other examples of such current research.

In this study, we find novel generating functions that conform to Truesdell's ascending and descending Fq-equation, and we also deduce several identities for the q-analogue of the gamma function. We have extended the F-equation to its q-analogue, the Fq-equation, in order to do this. We have also identified a range of I-functions, both ascending and descending, that meet Truesdell's Fq-equation. We have used these different structures to derive many I-function generating functions. The outputs are verified by obtaining certain specific instances of these results, which also provide some new results.

LITERATURE REVIEW

Choi, Junesang et.al (2018). Combinatorics, astronomy, applied mathematics, physics, and engineering are just few of the many scientific fields that have made use of special functions for centuries. This Special Issue's principal objective is to offer a forum for exploring the possible applications of recently developed concepts and formulas for special functions. Numbers and polynomials associated with classical functions like the Bernoulli number, the Euler number, and the gamma function; gamma functions, beta functions, multiple gamma functions, and their q-extensions and other extensions; confluent hypergeometric functions; hypergeometric functions; generalized hypergeometric functions; multiple hypergeometric functions; and their various extensions and associated polynomials.

Du, Wei-Shih et.al (2021). This survey article's primary objective is to provide a synthesis of the existing literature on vector-valued virtually periodic functions and their applications. We take into account virtually periodic functions in two or more real variables independently from those in one real variable. We discuss various unanswered questions and avenues for further research on virtually periodic functions, citing almost two hundred sources.

Zeraoulia, Rafik. (2021). My advisor, Dr. Brahimi Mahmoud, from EDP, University of Batna 2, Algeria, oversaw my work for my Master's thesis. This study expands upon my previous work on the advection equation and its use in contemporary physics, which may be found.

Khosravian-Arab, Hassan et.al (2017). This work introduces the generalized-tempered Bessel functions of the first- and second-kind, abbreviated GTBFs-1 and GTBFs-2, respectively, for the convex domain. The GTBFs-1 and GTBFs-2 are taken into account as two specific situations. Then, we demonstrate that these operators are self-adjoint on appropriate domains, establishing that these functions may be written as solutions to two linear differential operators. Orthogonality, completeness, fractional derivatives and integrals, recursive relations, asymptotic formulae, and other relevant aspects of these sets of functions are demonstrated in detail. At last, these operations are carried out to solve three real-world, fractional-order differential equations.

Kumar, Yazali et.al (2020). The purpose of this study is to develop a new generalized function, the Psi-function, that extends the functionality of the popular Fox H-function and I-function. Where did Saxena and Rathie get their I-function definition? We have also provided exceptional examples, basic features, and circumstances under which convergence occurs.

HANKEL TRANSFORM OF I-FUNCTION OF ONE VARIABLE

Here, we do an analysis of the one-variable Hankel transform of an I-function and identify the one-variable Hankel transform of an H-function as a special instance of the former.

One-variable I-function Hankel transform =

$$\mathcal{H}_\nu\left\{\left(\frac{\rho}{x}\right)^{1/2} I^{m,n}_{p_i,q_i:r}\left[\alpha x^\sigma \left|\begin{matrix}T\\T'\end{matrix}\right.\right]\right\}$$

$$= \frac{2^{1/2}}{\rho} I^{m,n+1}_{p_i+2,q_i:r}\left[\alpha\left(\frac{2}{\rho}\right)^\sigma \left|\begin{matrix}\left(\frac{1}{4}-\frac{\nu}{2},\frac{\sigma}{2}\right),T,\left(\frac{1}{4}+\frac{\nu}{2},\frac{\sigma}{2}\right)\\ T'\end{matrix}\right.\right]$$

where $\rho,\nu,\alpha \in \boldsymbol{C};\ \sigma>0,\rho>0,\alpha>0$ satisfy the condition

$$\Re(\rho)+\Re(\nu)+\sigma\, min_{1\le j\le m}\left[\frac{\Re(b_j)}{\beta_j}\right]>-\frac{3}{2}\ ;\ \Re(\rho)+\sigma\, max_{1\le j\le n}\left[\frac{1-\Re(a_j)}{\alpha_j}\right]<-\frac{1}{2}$$

Proof: By plugging in the left-hand side of the equation representing the Hankel transform definition, we obtain

$$\mathcal{H}_\nu\left\{\left(\frac{\rho}{x}\right)^{1/2} I^{m,n}_{p_i,q_i:r}\left[\alpha x^\sigma \left|\begin{matrix}T\\T'\end{matrix}\right.\right]\right\} = \int_0^\infty x\, J_\nu(\rho x)\left\{\left(\frac{\rho}{x}\right)^{1/2} I^{m,n}_{p_i,q_i:r}\left[\alpha x^\sigma \left|\begin{matrix}T\\T'\end{matrix}\right.\right]\right\}dx$$

The right-hand side of a contour integral of the Mellin-Barne type may be used to define the I-function of a single variable as follows:

$$\int_0^\infty (\rho x)^{1/2} J_\nu(\rho x)\left\{\frac{1}{2\pi\omega}\int_L \theta(\xi)\,\alpha^\xi x^{\sigma\xi} d\xi\right\}dx$$

Altering the allowed sequence of integration under the right-hand side criteria yields

$$\frac{1}{2\pi\omega}\int_L \theta(\xi)\alpha^\xi\left\{\int_0^\infty (\rho x)^{1/2} J_\nu(\rho x)\, x^{\sigma\xi} dx\right\} d\xi$$

using the result replacing μ by $\sigma\xi$ The right-hand-side integral of has the following form on the inside

$$\frac{1}{2\pi\omega}\int_L \theta(\xi)\alpha^\xi 2^{\sigma\xi+\frac{1}{2}}\rho^{-\sigma\xi-1}\frac{\Gamma\left(\frac{\sigma\xi}{2}+\frac{\nu}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu}{2}-\frac{\sigma\xi}{2}+\frac{1}{4}\right)}\ d\xi$$

by shuffling the letters around, we get

$$\frac{2^{\frac{1}{2}}}{\rho}\frac{1}{2\pi\omega}\int_L \theta(\xi)\alpha^{\xi}\left(\frac{2}{\rho}\right)^{\sigma\xi}\frac{\Gamma\left(1-\left(\frac{1}{4}-\frac{\nu}{2}\right)+\frac{\sigma\xi}{2}\right)}{\Gamma\left(\left(\frac{1}{4}+\frac{\nu}{2}\right)-\frac{\sigma\xi}{2}\right)}\,d\xi$$

The right-hand side of the equation is found by writing the contour integral of Mellin-Barne's type as an I-function of one variable.

Substituting $\nu = 0$, The zero-order Hankel transform is found by solving:

$$\mathcal{H}_0\left\{\left(\frac{\rho}{x}\right)^{1/2} I_{p_i,q_i:r}^{m,n}\left[\alpha x^{\sigma}\left|\begin{matrix}T\\T'\end{matrix}\right.\right]\right\}$$

$$=\frac{2^{1/2}}{\rho} I_{p_i+2,q_i:r}^{m,n+1}\left[\alpha\left(\frac{2}{\rho}\right)^{\sigma}\left|\begin{matrix}\left(\frac{1}{4},\frac{\sigma}{2}\right),T,\left(\frac{1}{4},\frac{\sigma}{2}\right)\\T'\end{matrix}\right.\right]$$

Where

$\rho,\nu,\alpha\ \in \boldsymbol{C};\ \ \sigma>0,\rho>0,\alpha>0$ satisfy the condition

$$\Re(\rho)+\sigma\min_{1\le j\le m}\left[\frac{\Re(b_j)}{\beta_j}\right]>-\frac{3}{2}\quad;\quad \Re(\rho)+\sigma\max_{1\le j\le n}\left[\frac{1-\Re(a_j)}{\alpha_j}\right]<-\frac{1}{2}$$

AXISYMMETRIC DIRICHLET POTENTIAL PROBLEM

For example, in the book of Andrews, Larry C., and Shivamoggi Bhimsen K., the Axisymmetric Dirichlet problem for a half space is mathematically defined as

$$u_{xx}+\frac{1}{x}u_x+u_{zz}=0,\quad 0<x<\infty,\ z>0$$

boundary conditions are

$$u(x,0)=f(x),\qquad 0<x<\infty$$

$$u(x,z)\to 0\ \text{ as }\ \sqrt{(x^2+z^2)}\to\infty, z>0$$

The transformed issue looks like this once the x-variable is subjected to a Hankel transform of order zero:

$$U_{zz}-\rho^2 U=0\,,\qquad z>0$$

Having Boundaries

$$U(\rho,0)=F(\rho)$$

$$U(\rho,z)\to 0\qquad as\quad z\to\infty$$

Where

$$\mathcal{H}_0\{u(x,z);x\to\rho\}=U(\rho,z)$$

$$\mathcal{H}_0\{f(x);\rho\}=F(\rho)$$

clearly, the solution of is

$$U(\rho,z)=F(\rho)e^{-\rho z}$$

Using the Hankel inversion formula for integration, we obtain

$$u(x,z)=\mathcal{H}_0^{-1}[F(\rho)e^{-\rho z};\rho\to x]=\int_0^{\infty}\rho F(\rho)e^{-\rho z}J_0(\rho x)d\rho$$

substituting $f(x)=\left(\frac{\rho}{x}\right)^{1/2} I_{p_i,q_i:r}^{m,n}\left[\alpha x^{\sigma}\left|\begin{matrix}T\\T'\end{matrix}\right.\right]$ The answer to the Axisymmetric Dirichlet problem for a half space is given in the following equation:

$$u(x,z)=\frac{2^{1/2}}{z}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(-\frac{x^2}{z^2}\right)^k$$

$$\times I_{p_i+4,q_i+3:r}^{m+3,n+1}\left[\alpha(2z)^{\sigma}\left|\begin{matrix}\left(\frac{1}{4},\frac{\sigma}{2}\right),T,\left(\frac{1}{4},\frac{\sigma}{2}\right),\left(\frac{1}{2},\frac{\sigma}{2}\right),\left(1,\frac{\sigma}{2}\right)\\ \left(\frac{1}{2}+k,\frac{\sigma}{2}\right),\left(1+k,\frac{\sigma}{2}\right),(1,\sigma),T'\end{matrix}\right.\right]$$

where
$\rho,\alpha\ \in \boldsymbol{C};\ \ \sigma>0,\rho>0,\alpha>0;\Re(z)>|Im(x)|;\ \ \Re(1-\sigma\xi)>0$
satisfy the conditions

$$\sigma\min_{1\le j\le m}\left[\frac{\Re(b_j)}{\beta_j}\right]>-\frac{3}{2}\quad;\quad \sigma\max_{1\le j\le n}\left[\frac{1-\Re(a_j)}{\alpha_j}\right]<-\frac{1}{2}$$

Proof: When the Hankel transform of is applied to (x) , we have

$$\mathcal{H}_0\{f(x);\rho\}=\frac{2^{1/2}}{\rho} I_{p_i+2,q_i:r}^{m,n+1}\left[\alpha\left(\frac{2}{\rho}\right)^{\sigma}\left|\begin{matrix}\left(\frac{1}{4},\frac{\sigma}{2}\right),T,\left(\frac{1}{4},\frac{\sigma}{2}\right)\\T'\end{matrix}\right.\right]=F(\rho)$$

substituting the value of $F(\rho)$ solution gives

$$u(x,z)=\int_0^{\infty}\rho\left\{\frac{2^{1/2}}{\rho} I_{p_i+2,q_i:r}^{m,n+1}\left[\alpha\left(\frac{2}{\rho}\right)^{\sigma}\left|\begin{matrix}\left(\frac{1}{4},\frac{\sigma}{2}\right),T,\left(\frac{1}{4},\frac{\sigma}{2}\right)\\T'\end{matrix}\right.\right]\right\}e^{-\rho z}J_0(\rho x)d\rho$$

We get the contour integral of the one-variable I-function in terms of the Mellin-Barne type formula.

$$u(x,z)=2^{1/2}\int_0^{\infty}\left\{\frac{1}{2\pi\omega}\int_L \alpha^{\xi}\theta'(\xi)\left(\frac{2}{\rho}\right)^{\sigma\xi}d\xi\right\}e^{-\rho z}J_0(\rho x)d\rho$$

By rearranging the allowed steps of integration under the given constraints, we get

$$u(x,z)=2^{1/2}\frac{1}{2\pi\omega}\int_L \alpha^{\xi}\theta'(\xi)2^{\sigma\xi}\left\{\int_0^{\infty}\rho^{-\sigma\xi}e^{-\rho z}J_0(\rho x)d\rho\right\}d\xi$$

Taking the right-hand-side solution of Eq.we get

$$u(x,z)=2^{1/2}\frac{1}{2\pi\omega}\int_L \alpha^{\xi}\theta'(\xi)2^{\sigma\xi}\left\{\frac{\Gamma(1-\sigma\xi)}{z^{(1-\sigma\xi)}}\ {}_2F_1\left[\frac{(1-\sigma\xi)}{2},\frac{(2-\sigma\xi)}{2};1;-\frac{x^2}{z^2}\right]\right\}d\xi$$

We have an enlargement of the Hypergeometric function.

$$u(x,z)=\frac{2^{1/2}}{z}\frac{1}{2\pi\omega}\int_L \alpha^{\xi}\theta'(\xi)2^{\sigma\xi}\left\{\frac{\Gamma(1-\sigma\xi)}{z^{(-\sigma\xi)}}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}-\frac{\sigma\xi}{2}\right)_k\left(1-\frac{\sigma\xi}{2}\right)_k}{(1)_k k!}\left(-\frac{x^2}{z^2}\right)^k\right\}d\xi$$

$$u(x,z)=\frac{2^{1/2}}{z}\frac{1}{2\pi\omega}\int_L \alpha^{\xi}\theta'(\xi)(2z)^{\sigma\xi}\Gamma(1-\sigma\xi)$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}-\frac{\sigma\xi}{2}+k\right)\Gamma\left(1-\frac{\sigma\xi}{2}+k\right)}{\Gamma\left(\frac{1}{2}-\frac{\sigma\xi}{2}\right)\Gamma\left(1-\frac{\sigma\xi}{2}\right)}\frac{1}{(k!)^2}\left(-\frac{x^2}{z^2}\right)^k d\xi$$

rearranging the terms, we get

$$u(x,z)=\frac{2^{1/2}}{z}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(-\frac{x^2}{z^2}\right)^k\frac{1}{2\pi\omega}\int_L \alpha^{\xi}\theta'(\xi)(2z)^{\sigma\xi}$$

$$\times\frac{\Gamma\left(\frac{1}{2}-\frac{\sigma\xi}{2}+k\right)\Gamma\left(1-\frac{\sigma\xi}{2}+k\right)\Gamma(1-\sigma\xi)}{\Gamma\left(\frac{1}{2}-\frac{\sigma\xi}{2}\right)\Gamma\left(1-\frac{\sigma\xi}{2}\right)}d\xi$$

The right-hand side of the equation is found by writing the Mellin-Barne's contour integral as an I-function of a single variable.

SPECIAL CASES

Substituting $r=1$ The I-function of one variable is equivalent to Fox's H-function of one variable as shown in with the same assumptions. $a_{j1},\alpha_{j1},b_{j1},\beta_{j1}$ as a_j,α_j,b_j,β_j respectively. This gives us the Hankel transform of order zero for a one-variable H-function.

$$\mathcal{H}_0\left\{\left(\frac{\rho}{x}\right)^{1/2}H_{p,q}^{m,n}\left[\alpha x^{\sigma}\left|\begin{matrix}U\\U'\end{matrix}\right.\right]\right\}=\frac{2^{1/2}}{\rho}H_{p+2,q}^{m,n+1}\left[\alpha\left(\frac{2}{\rho}\right)^{\sigma}\right|\left(\frac{1}{4}\right.$$

Where

$\rho,\nu,\alpha\ \in \boldsymbol{C};\ \ \sigma>0,\rho>0,\alpha>0$ Obtain the desired result

$$\Re(\rho)+\sigma\min_{1\le j\le m}\left[\frac{\Re(b_j)}{\beta_j}\right]>-\frac{3}{2}\ ;\ \ \Re(\rho)+\sigma\ m$$

CONCLUSION

Our goal is to use a Hankel transform of order zero to get the answer to a boundary value issue using cylindrical coordinates. Boundary value problems and their solutions using the Fourier Cosine transform of the inverse function of one variable, the Hankel transform of the inverse function of one variable, and the Hankel transform of the inverse function of one variable are central to the study of mathematical physics.

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