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# On Lamb Waves in a Thermoelastic Plate in the Presence of IdealFluid with Varied Temperature

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***Abstract – In this paper,we studied the phenomenon of wave motion in a homogeneously isotropic, thermoelastic solid plate framed with ideal fluid layers on its both sides with varying temperatures. In the light of the classical theory of thermo-elasticity, all the work is carried out and for Lamb-type thermoelastic wavespropagating in the plate,the secular equations are gobbled up for symmetric and skew-symmetric wave style. The different cases of secular equations are also discussed in the framework of the uncoupled thermo-elasticity.The dispersion equations for three different regions are also deduced. It is found that the SH mode remains unaffected by thermal variations and keeps itself isolated from the rest of the coupled motion of elastic waves (longitudinal and SV modes) and thermal waves (T-mode).Onewave in each liquid layer also exists due to the presence of ideal fluid loadings. The numerical resultsfor an aluminum-epoxy materialcladded with water are carried out. A wide range of scopes of this research area is available in various fields such as ultrasonics, earthquake engineering, soil dynamics, seismology, etc.***

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## 1. INTRODUCTION

The elastic waves are used to measure defects and elastic properties in solid materials haveestablished great attention, and various important applications have been developed recently. Inthe non-destructive evaluation of hard materials, the coupling of elastic waves with liquid-loaded materials has appeared as animportant field. The acoustic waves thatare replicated from the solid-liquid interface have a lot of information regarding the solid structure properties together with the presence of the internal defect, interface quality etc. In deformable-body temperature varies from point to point and with time. This temperature variation is due to the deformation process and exchange of heat with the external medium in which mechanical energy is changed into heat energy. The degradation in thermo-elastic energy results in the damping of elastic body vibrations. Firstly,Lamb [1] developed the theory of waves known as Lamb waves**.**The Lamb waves are created on the belief that when the solid plate is cladded with the liquid it varies the amplitude and propagation velocity of the Lamb waves in the free boundaries due to viscous and inertial effects of the fluid in the plate. Theconsequence of thefluid on the Lamb wavespropagation in a plate of fixed-widthsandwiched between homogeneous liquidhalf-space on its both sides investigated by Schoch[2]and found that some amount ofenergyin the plate is attached with the fluid in the form of radiation, while the most of energy leftovers in the solid. Forplane thermoelastic and magnetothermoelastic waves, an exact solution of the secular equation is derived by Puri [3] and obtained the solutions using approximate expansions for low and high frequencies and small coupling. Plona et. al. [4]investigated Lamb and Rayleigh waves at solid-liquid boundaries and derived that the simple Lamb or Rayleigh mode approach gives unexpected results when the solid and liquid densities are nearly equal. The influence of fluid layers on Lamb wavespropagation in asolidplate is discussed by Wu and Zhu[6]and obtained the frequency equation for the same.

Exposedto isothermal and insulated conditions,thethermally conducting elastic waves for a rigidly fixed and stress-free homogeneous and an isotropicmaterial in the light of non-classical theories of thermo-elasticity are studied by Sharma et.al. [7]. Thetime-harmonic Lamb waves propagation in the thermo-elasticmaterial fringed with non-viscousfluid loading on the bottom and top of the plate is investigated by Sharma and Pathania [8]. Sharma and Pathania [9]studied the motion of Rayleigh and Lamb waves in thermally conducting elastic plate cladded with homogeneousfluid coatingsor half-spaceson its both borders in the framework of the non-classical theory of thermo-elasticity. They showed that the shear horizontal component of waves decouples from the primary stream of wave motion and evaluated the frequency equations for non-leaky Lamb waves, leaky Rayleigh waves and leaky Lamb waves.Propagation of wave in a liquid saturated porous solid with micro-polar elastic skelton at the boundary surface has been evaluated by [10, 11]. Pathania et. al. [12] investigated the thermoelastic waves in anisotropic plates immersed in viscous liquid layers innon-classical theories of thermo-elasticity. Kumar et.al. [13] discussed the circular crested and straight wave motion in micro-stretch thermoelastic plate surroundedbynon-viscousfluid coatingson both sides with varying temperatures. Pathania et. al. [14] also studied the characteristics of the circular waves in a homogeneous and transversely isotropicthermo-elastic materialsurroundedby conducting viscous fluid loading layers (or halfspaces) on the top and bottom of the plate. Recently, Barak et al. [15] evaluated the reflection and refraction of wave in two welded contact infinite unbounded half-spaces and the effect of the loosely bounded interface on wave propagation between two half-space has been obtained by Barak and Kaliraman [16].

Here we analyze the motion of Lamb waves in athermally conducting elastic homogeneousand isotropic plate in the presence of an ideal fluid layer on its both sides at varying temperatures. The governing equations are solved in the x-z plane and it is found that there exist three coupled waves in the solid plate and one wave in each liquid layer. For Lamb waves, the frequency equation issolved analyticallyforthe classical theory of thermo-elasticity and further deduced for uncoupled thermoelasticity. The equations for various regions have been deduced from the secular equation depending upon the type of characteristic roots. The aluminum-epoxy composite materialis selected for the solid plate and water is taken as a fluid to carry out the numerical results. The mathematical and graphical results are closely related to each other.

## 2. FORMULATION OF THE PROBLEM

We consider a thermally conducting elastic homogeneous and isotropic solidmaterialhavingthe thickness $2d$. Initially, the solid plate is at an undisturbed state and unvarying temperature $T_0$. The solid plate is cladded with a homogeneous ideal fluid of thickness $h$ on both borders i.e.on top and the bottom. It is supposed that from the interlayers of the liquid medium no reflection takes place. The origin of the cartesian coordinate system $o-xyz$ is taken at any point in the center of the plate. The wave propagates along the $x$ - direction andthe field extentsremain

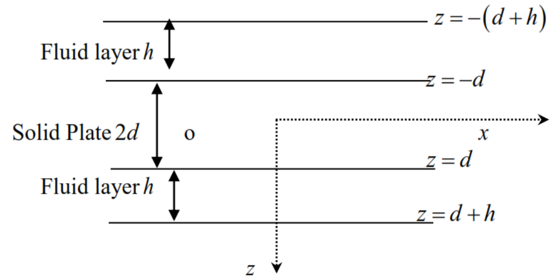



**Figure-1: Geometry of the Problem**

explicitly independent of $y$ -coordinate which implies $\partial/\partial y=0,$ but depend implicitly on $y$ -coordinate such that the shear component of displacement is non-zero**.** The z-axis points to a vertically downwards direction alongthe plate thicknessas illustrated in Figure-1.

The constitutive relationsand fundamental equations in the light of the classical theory of thermo-elasticity for the plate in the non-attendance ofbody forces andheat sources[8] are,

$$\mu\nabla^2\vec{u}+(\lambda+\mu)\nabla\nabla.\vec{u}-\beta\nabla T=\rho\ddot{\vec{u}} \tag{1}$$

$$K\nabla^2 T-\rho C_e\dot{T}=\beta T_0\nabla.\dot{\vec{u}} \tag{2}$$

The governing equations and temperature relation in the non-attendance of heat sources and body forcesforthe fluid medium [13],are given by

$$\lambda_L\nabla\nabla.\vec{u}_{L_i}-\beta^*\nabla T_L=\rho_L\ddot{\vec{u}}_{L_i}\,;\, i=1,\,2 \tag{3}$$

$$K^*\nabla^2 T_L-\rho_L c_L\dot{T}_L=\beta^* T_0^*\nabla.\dot{\vec{u}}_{L_i}\,;\, i=1,\,2 \tag{4}$$

Where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}.$$

Here the dot over a symbol represents the time derivative and the comma in the subscript denotes the spatial derivative. $\lambda$ , $\mu$ are Lame's parameters, $\rho$ is the density of solid material, $\vec{u}(x,z,t)=(u,0,w)$ is the displacement vector, $C_e$ denotes the specific heat at a constant strain of the solid material, $T(x,z,t)$ is temperature changeand*K*is the coefficient of thermal conductivity.Also $\beta=(3\lambda+2\mu)\alpha_T$, where $\alpha_T$ is the linear thermal expansion. In the same manner, for the liquid, $\vec{u}_{L_i}(x,z,t)=(u_{L_i},0,w_{L_i})$ is denoting the displacement vector in fluid, $\rho_L$ is the density and $\lambda_L$ is the bulk modulus of liquid respectively. Fromthe ambient temperature $T_0^*$, $T_L$ is the deviation in temperature of the fluid and $K^*$ is the thermal conductivity in fluid layers. Also $\beta^* = 3\lambda_L \alpha^*$, where $\alpha^*$ is the coefficient of volume thermal expansion. The subscript $i$ represents that liquid takes the value 1 and 2 for the bottom and top fluid layers. Here $\nabla$ and $\nabla^2$ is the Nabla and Laplacian operator respectively.

Consider the dimensionless quantities as

$$x' = \frac{\omega^* x}{c_1}, z' = \frac{\omega^* z}{c_1}, u' = \frac{\rho\omega^* c_1 u}{\beta T_0}, w' = \frac{\rho\omega^* c_1 w}{\beta T_0}, u_{L_i}{}' = \frac{\rho\omega^* c_1 u_{L_i}}{\beta T_0}, w_{L_i}{}' = \frac{\rho\omega^* c_1 w_{L_i}}{\beta T_0}, t' = \omega^* t,$$
$$T' = \frac{T}{T_0}, T_L' = \frac{T_L}{T_0}. \qquad (5)$$

Making use of quantities (5), the dimensionlessfundamental equations of motion and energyequation in the solid plate and liquid layers, after suppressing primes, are

$$\delta^2 \nabla^2 \vec{u} + \left(1-\delta^2\right)\nabla\nabla.\vec{u} - \nabla T = \ddot{\vec{u}} \qquad (6)$$

$$\nabla^2 T - \dot{T} = \varepsilon_T \nabla.\dot{\vec{u}} \qquad (7)$$

$$\nabla\nabla.\vec{u}_{L_i} - \frac{\rho c_1^2 \bar{\beta}}{\lambda_L}\nabla T_L = \frac{1}{\delta_L^2}\ddot{\vec{u}}_{L_i};\ i = 1, 2 \qquad (8)$$

$$\dot{T}_L = \frac{\varepsilon_L \lambda_L}{\rho c_1^2 \bar{\beta}}\nabla.\dot{\vec{u}}_{L_i};\ i = 1, 2 \qquad (9)$$

where

$$c_L{}^2 = \frac{\lambda_L}{\rho_L}, \delta^2 = \frac{c_2^2}{c_1^2}, \delta_L{}^2 = \frac{c_L{}^2}{c_1{}^2}, c_1^2 = \frac{\lambda+2\mu}{\rho}, c_2^2 = \frac{\mu}{\rho}, \omega^* = \frac{C_e(\lambda+2\mu)}{K}, \varepsilon_T = \frac{\beta^2 T_0}{\rho C_e(\lambda+2\mu)},$$
$$\varepsilon_L = \frac{\beta^{*2} T_0^*}{\rho_L C_v^* \lambda_L}, \bar{\beta} = \frac{\beta^*}{\beta}. \qquad (10)$$

Here $\omega^*$ is the characteristic frequency of the material plate and $C_v^*$ denotes the specific heat of the fluid at constant volume.Inthe solid plate $c_1, c_2$ denote the longitudinal and shear wave velocities and $c_L$ denotes the velocity of sound in the liquid,. Also $\varepsilon_T$ and $\varepsilon_L$ are the thermomechanical coupling constants in the material plate and fluid layers respectively.

To solve the above equations, we consider

$$u = \varphi_{s,x} + \psi_{s,z}, \quad w = \varphi_{s,z} - \psi_{s,x} \qquad (11)$$

where $\varphi_s, \psi_s$ represents the potential functions for the longitudinal and shear waves. In the inviscid fluid layers,the shear motion does not exist, thusin the absence of shear motioninthe fluid we get

$$u_{L_i} = \varphi_{L_i,x}, \quad w_{L_i} = \varphi_{L_i,z};\ i = 1,2 \qquad (12)$$

where $\varphi_{L_i}$ is the scalar velocity potential for the bottom and top fluid layers $i = 1, 2.$

Plugging equations (11) and (12) in equations (6)-(9), we obtain the potential functions $\varphi_s, \psi_s, \varphi_{L_i}$, temperatures $T$ and $T_L$ as

$$\nabla^2 \psi_s - \frac{1}{\delta^2}\ddot{\psi}_s = 0 \qquad (13)$$

$$\nabla^2 \varphi_s - \ddot{\varphi}_s = T \qquad (14)$$

$$\nabla^2 T - \dot{T} = \varepsilon_T \nabla^2 \dot{\varphi}_s \qquad (15)$$

$$\left(1+\varepsilon_L\right)\nabla^2 \varphi_{L_i} - \frac{1}{\delta_L^2}\ddot{\varphi}_{L_i} = 0,\ \ i = 1,2 \qquad (16)$$

$$T_{L_i} = -\frac{\varepsilon_L \rho_L}{\rho\bar{\beta}\left(1+\varepsilon_L\right)}\ddot{\varphi}_{L_i},\ \ i = 1,2 \qquad (17)$$

## 3. SOLUTION OF THE PROBLEM

Since the waves are propagating along the x-axis in the positive direction ofthe thermo-elastic plate, we take the solutions as

$$\left\{\varphi_s, \psi_s, T, \varphi_{L_1}, \varphi_{L_2}, T_{L_1}, T_{L_2}\right\} = \left\{\bar{\varphi}_s(z), \bar{\psi}_s(z), \bar{T}(z), \bar{\varphi}_{L_1}(z), \bar{\varphi}_{L_2}(z), \bar{T}_{L_1}(z), \bar{T}_{L_2}(z)\right\} e^{i\xi(x-ct)} \qquad (18)$$

Here $c = \omega/\xi$, $c$, $\xi$ and $\omega$ represent the dimensionless phase velocity, wavenumber andangular frequency of plane waves.

Invoking the solutions (18) in equations (13)-(17), the expressions for $\varphi_s, \psi_s, T, T_L, \varphi_{L_1}$ and $\varphi_{L_2}$ are attained as

$$\left.\begin{aligned} \varphi_s &= \sum_{i=1}^{2}\left(A_i \sin m_i z + B_i \cos m_i z\right) e^{i\xi(x-ct)} \\ T &= \sum_{i=1}^{2}\left(\alpha^2 - m_i^2\right)\left(A_i \sin m_i z + B_i \cos m_i z\right) e^{i\xi(x-ct)} \\ \psi_s &= \left(A_3 \sin m_3 z + B_3 \cos m_3 z\right) e^{i\xi(x-ct)} \end{aligned}\right\} -d < z < d \qquad (19)$$

$$\left.\begin{aligned} \varphi_{L_1} &= A_4 \sin m_4\left[z-(d+h)\right] e^{i\xi(x-ct)} \\ T_{L_1} &= S_L A_4 \sin m_4\left[z-(d+h)\right] e^{i\xi(x-ct)} \end{aligned}\right\} d < z < d+h \qquad (20)$$

$$\left.\begin{aligned} \varphi_{L_2} &= A_5 \sin m_4\left[z+d+h\right] e^{i\xi(x-ct)} \\ T_{L_2} &= S_L A_5 \sin m_4\left[z+d+h\right] e^{i\xi(x-ct)} \end{aligned}\right\} -(d+h) < z < -d \qquad (21)$$

Where

$$\alpha^2=\xi^2(c^2-1),\ m_i^2=\xi^2\left(a_i^2c^2-1\right), i=1,2,\ m_3^2=\xi^2\left(\frac{c^2}{\delta^2}-1\right), m_4^2=\xi^2\left(\frac{c^2}{\delta_L^2(1+\varepsilon_L)}-1\right) \tag{22}$$

$$S_L=\frac{\varepsilon_L\rho_L\omega^2}{\rho\bar{\beta}(1+\varepsilon_L)},\ a_1^2,a_2^2=\frac{1}{2}\left\{\left(1+(1+\varepsilon_T)i\omega^{-1}\right)\pm\left[\left(1-(1-\varepsilon_T)i\omega^{-1}\right)^2-4\varepsilon_T\omega^{-2}\right]^{\frac{1}{2}}\right\} \tag{23}$$

Puri [3] firstly obtained the expressions for $a_1^2, a_2^2$. Theacoustical pressure vanishes at external boundaries $z=\pm(d+h)$ to ensure a bounded solution due to the chosen potential functions $\varphi_{L_1}$ and $\varphi_{L_2}$. Here $\varphi_{L_1}$ and $\varphi_{L_2}$ are the solutions of the standing wave and they are solutions of traveling waves in leaky Lamb waves case.

## 4. BOUNDARY CONDITIONS

The stress traction, displacement and heat flux at the solid-fluid interfaces $z=\pm\ d$ may be written as:

(i) For the solid plate, the dimensionless normal element of the stress tensor must be equal to the pressure of the fluid, i.e.

$$\ddot{\varphi}_s-2\delta^2\left(\varphi_{s,xx}+\psi_{s,xz}\right)=\frac{\omega^2\rho_L}{\rho}\varphi_{L_i},\qquad i=1,2 \tag{24}$$

(ii) The dimensionless shearelement of the stress tensor must be zero, i.e.

$$2\varphi_{s,xz}+\psi_{s,zz}-\psi_{s,xx}=0 \tag{25}$$

(iii) The dimensionless normal displacement elementof the plate must be equal to thefluid, i.e.

$$\varphi_{s,z}-\psi_{s,x}=\varphi_{L_i,z};\ i=1,2 \tag{26}$$

(iv) The boundary condition for thermal case is specified by

$$T,_z+H(T-T_L)=0 \tag{27}$$

where H is the coefficient of heat transfer.

## 5. DERIVATION OF SECULAR EQUATION

Plugging therequired interface conditions (24)-(27) at $z=\pm d$, we obtain anarrangement of eight homogeneous linear equations with eight unknown amplitudes. For the existence ofa non-zero solution of the system of equations, the determinant of the coefficient matrix of these parameters is zero. Theprocedure used by Sharma and Pathania [8] aftersome arithmetical calculations of the determining factortogether with conditions $m_4\neq 0$ and $m_4h\neq(2n-1)\frac{\pi}{2},\ n=1,2,3...$ the dispersion relation for Lamb type waves with varying temperature yields

$$\left[\frac{T_1}{T_3}\right]^{\pm1}-\frac{m_1(\alpha^2-m_1^2)}{m_2(\alpha^2-m_2^2)}\left[\frac{T_2}{T_3}\right]^{\pm1}+\frac{\rho_L\omega^2m_1\left(m_3^2+\xi^2\right)\left(m_1^2-m_2^2\right)}{\rho\delta^2m_4\left(\xi^2-m_3^2\right)^2\left(\alpha^2-m_2^2\right)}\frac{T_4}{[T_3]^{\pm1}}+\frac{4\xi^2m_1m_3H}{\left(\xi^2-m_3^2\right)^2m_2}\left[\frac{T_1T_2}{T_3}\right]^{\pm1}$$
$$\left\{\left(1+\frac{\rho_L\omega^2(m_3^2+\xi^2)}{4\rho\delta^2\xi^2m_3m_4}\frac{T_4}{[T_3]^{\pm1}}\right)\left(\left[\frac{T_1}{T_3}\right]^{\mp1}-\frac{m_2\left(\alpha^2-m_1^2\right)}{m_1\left(\alpha^2-m_2^2\right)}\left[\frac{T_2}{T_3}\right]^{\mp1}\right)+\frac{\left(\xi^2-m_3^2\right)^2\left(m_1^2-m_2^2\right)}{4\xi^2m_1m_3\left(\alpha^2-m_2^2\right)}\right.$$
$$\left.+\frac{S_L(m_3^4-\xi^4)}{4\xi^2m_3m_4\left(\alpha^2-m_2^2\right)}\frac{T_4}{[T_3]^{\pm1}}\left(\left[\frac{T_1}{T_3}\right]^{\mp1}-\frac{m_2}{m_1}\left[\frac{T_2}{T_3}\right]^{\mp1}\right)\right\}=-\frac{4\xi^2m_1m_3\left(m_1^2-m_2^2\right)}{\left(\xi^2-m_3^2\right)^2\left(\alpha^2-m_2^2\right)} \tag{28}$$

where $T_i=\tan(m_id);\ i=1,2,3,\ \ T_4=\tan m_4h$.
The skew-symmetric mode corresponds to by superscripted + sign and symmetric mode corresponds to superscripted – sign.

If $\rho_L\rightarrow 0$, i.e. in the nonappearance of fluid layers, equation (28) becomes

$$\left[\frac{T_1}{T_3}\right]^{\pm1}-\frac{m_1\left(\alpha^2-m_1^2\right)}{m_2\left(\alpha^2-m_2^2\right)}\left[\frac{T_2}{T_3}\right]^{\pm1}+\frac{4\xi^2m_1m_3H}{\left(\xi^2-m_3^2\right)^2m_2}\left[\frac{T_1T_2}{T_3}\right]^{\pm1}\left\{\left[\left[\frac{T_1}{T_3}\right]^{\mp1}-\frac{m_2\left(\alpha^2-m_1^2\right)}{m_1\left(\alpha^2-m_2^2\right)}\left[\frac{T_2}{T_3}\right]^{\mp1}\right]\right.$$
$$\left.+\frac{\left(\xi^2-m_3^2\right)^2\left(m_1^2-m_2^2\right)}{4\xi^2m_1m_3\left(\alpha^2-m_2^2\right)}\right\}=-\frac{4\xi^2m_1m_3\left(m_1^2-m_2^2\right)}{\left(\xi^2-m_3^2\right)^2\left(\alpha^2-m_2^2\right)} \tag{29}$$

which represents the secular equation in a stress-free thermally conducting elastic solid for uniform temperature.

For a thermally insulated stress-free solid, the secular equation (29) can be obtained by setting $H\rightarrow 0$, thuswe have

$$\left[\frac{T_1}{T_3}\right]^{\pm1}-\frac{m_1\left(\alpha^2-m_1^2\right)}{m_2(\alpha^2-m_2^2)}\left[\frac{T_2}{T_3}\right]^{\pm1}=-\frac{4\xi^2m_1m_3\left(m_1^2-m_2^2\right)}{\left(\xi^2-m_3^2\right)^2\left(\alpha^2-m_2^2\right)} \tag{30}$$

and for an isothermal stress-free plate $H\rightarrow\infty$, thus we get

$$\left[\frac{T_1}{T_3}\right]^{\mp1}-\frac{m_2(\alpha^2-m_1^2)}{m_1\left(\alpha^2-m_2^2\right)}\left[\frac{T_2}{T_3}\right]^{\mp1}=-\frac{\left(\xi^2-m_3^2\right)^2\left(m_1^2-m_2^2\right)}{4\xi^2m_1m_3\left(\alpha^2-m_2^2\right)} \tag{31}$$

The frequency equations (30) and (31) represent the thermally insulated and isothermal cases for a stress-free thermoelastic plate and matches exactly withalready obtained results of Sharma et al. [7] and Sharma and Pathania [8].

## 6. DIFFERENT REGIONS OF THE SECULAR EQUATION

The equations (22) can be written as

$$\alpha^2=\xi^2\left(c^2-1\right)=\omega^2-\xi^2,\ m_i^2=\xi^2\left(a_i^2c^2-1\right)=a_i^2\omega^2-\xi^2, i=1,2,\ m_3^2=\xi^2\left(\frac{c^2}{\delta^2}-1\right)=\frac{\omega^2}{\delta^2}-\xi^2. \quad (32)$$

Here depending on whether $\xi^2 \geq \omega^2, \omega^2/\delta^2, a_1^2\omega^2, a_2^2\omega^2$ or $c^2 \leq 1, \delta^2, 1/a_1^2, 1/a_2^2$, we may have $\alpha, m_1, m_2, m_3$ to be imaginary, zero, or real. Thus the dispersion equation (28) is transformed as follows.

**RegionI:**For $\xi > \omega/\delta, \Rightarrow c<\delta, 1, 1/a_1, 1/a_2$ and accordingly, we get $\alpha = i\alpha'$, $m_i = im_i', \ i=1,2,3$. Therefore, equation (28) becomes

$$\left[\frac{\tanh m_1'd}{\tanh m_3'd}\right]^{\pm 1}-\frac{m_1'\left(\alpha'^2-m_1'^2\right)}{m_2'\left(\alpha'^2-m_2'^2\right)}\left[\frac{\tanh m_2'd}{\tanh m_3'd}\right]^{\pm 1}\pm\frac{\rho_L\omega^2 m_1'\left(\xi^2-m_3'^2\right)\left(m_1'^2-m_2'^2\right)}{\rho\delta^2 m_4\left(\xi^2+m_3'^2\right)^2\left(\alpha'^2-m_2'^2\right)}\frac{\tan m_4h}{\left[\tanh m_3'd\right]^{\pm 1}}$$
$$\mp\frac{4\xi^2 m_1'm_3'H}{\left(\xi^2+m_3'^2\right)^2 m_2'}\left[\frac{\tanh m_1'd\tanh m_2'd}{\tanh m_3'd}\right]^{\pm 1}\left\{\left[1\mp\frac{\rho_L\omega^2\left(\xi^2-m_3'^2\right)}{4\rho\xi^2\delta^2 m_3'm_4}\frac{\tan m_4h}{\left[\tanh m_3'd\right]^{\pm 1}}\right]\left[\left[\frac{\tanh m_1'd}{\tanh m_3'd}\right]^{\mp 1}\right.\right.$$
$$\left.-\frac{m_2'\left(\alpha'^2-m_1'^2\right)}{m_1'\left(\alpha'^2-m_2'^2\right)}\left[\frac{\tanh m_2'd}{\tanh m_3'd}\right]^{\mp 1}\right]-\frac{\left(\xi^2+m_3'^2\right)^2\left(m_1'^2-m_2'^2\right)}{4\xi^2 m_1'm_3'\left(\alpha'^2-m_2'^2\right)}+\frac{S_L(m_3'^4-\xi^4)}{4\xi^2 m_3'm_4\left(\alpha^2-m_2'^2\right)}\frac{\tanh m_4h}{\left[\tanh m_3'd\right]^{\pm 1}}\times$$
$$\left.\left(\left[\frac{\tanh m_1'd}{\tanh m_3'd}\right]^{\mp 1}-\frac{m_2'}{m_1'}\left[\frac{\tanh m_2'd}{\tanh m_3'd}\right]^{\mp 1}\right)\right\}=\frac{4\xi^2 m_1'm_3'\left(m_1'^2-m_2'^2\right)}{\left(\xi^2+m_3'^2\right)^2\left(\alpha'^2-m_2'^2\right)} \quad (33)$$

**Region II:** For $\omega/\delta > \xi > \omega \Rightarrow \delta < c < 1$ and the secular equation (28) yields

$$\left[\frac{\tanh m_1'd}{\tan m_3d}\right]^{\pm 1}-\frac{m_1'\left(\alpha'^2-m_1'^2\right)}{m_2'\left(\alpha'^2-m_2'^2\right)}\left[\frac{\tanh m_2'd}{\tan m_3d}\right]^{\pm 1}\pm\frac{\rho_L\omega^2 m_1'\left(\xi^2+m_3^2\right)\left(\alpha_1^2-m_2'^2\right)}{\rho\delta^2 m_4\left(\xi^2-m_3^2\right)^2\left(\alpha'^2-m_2'^2\right)}\frac{\tan m_4h}{\left[\tan m_3d\right]^{\pm 1}}$$
$$+\frac{4\xi^2 m_1'm_3H}{\left(\xi^2-m_3^2\right)^2 m_2'}\left[\frac{\tanh m_1'd\tanh m_2'd}{\tan m_3d}\right]^{\pm 1}\left\{\left[1+\frac{\rho_L\omega^2\left(\xi^2+m_3^2\right)}{4\rho\delta^2\xi^2 m_3m_4}\frac{\tan m_4h}{\left[\tan m_3d\right]^{\pm 1}}\right]\left[\left[\frac{\tanh m_1'd}{\tan m_3d}\right]^{\mp 1}\right.\right.$$
$$\left.-\frac{m_2'\left(\alpha'^2-m_1'^2\right)}{m_1'\left(\alpha'^2-m_2'^2\right)}\left[\frac{\tanh m_2'd}{\tan m_3d}\right]^{\mp 1}\right]\pm\frac{\left(\xi^2-m_3^2\right)^2\left(m_1'^2-m_2'^2\right)}{4\xi^2 m_1'm_3\left(\alpha'^2-m_2'^2\right)}+\frac{S_L(m_3^4-\xi^4)}{4\xi^2 m_3m_4\left(\alpha^2-m_2'^2\right)}\frac{\tanh m_4h}{\left[\tanh m_3d\right]^{\pm 1}}\times$$
$$\left.\left(\left[\frac{\tanh m_1'd}{\tanh m_3d}\right]^{\mp 1}-\frac{m_2'}{m_1'}\left[\frac{\tanh m_2'd}{\tanh m_3d}\right]^{\mp 1}\right)\right\}=\mp\frac{4\xi^2 m_1'm_3\left(m_1'^2-m_2'^2\right)}{\left(\xi^2-m_3^2\right)^2\left(\alpha'^2-m_2'^2\right)} \quad (34)$$

**Region III:** For $\xi < \omega \Rightarrow c > 1$ the dispersion equation remains the same as given by equation (28).

## 7. UNCOUPLED THERMOELASTICITY

For uncoupled thermo-elasticity $\varepsilon = 0, \Rightarrow a_1^2 = 1$, $a_2^2 = i\omega^{-1}$ thus $m_1^2 = \alpha^2,\ m_2^2 = \xi^2\left(i\omega^{-1}c^2-1\right)$. Therefore, the dispersion equations (28) yields

$$\left[\frac{T_1}{T_3}\right]^{\pm 1}+\frac{\rho_L\omega^2\alpha(m_3^2+\xi^2)}{\rho\delta^2 m_4(\xi^2-m_3^2)^2}\frac{T_4}{[T_3]^{\pm 1}}+\frac{4\alpha\xi^2 m_3H}{(\xi^2-m_3^2)^2 m_2}\left[\frac{T_1T_2}{T_3}\right]^{\pm 1}\left\{\left[\frac{T_1}{T_3}\right]^{\mp 1}\left[1+\frac{\rho_L\omega^2(m_3^2+\xi^2)}{4\rho\delta^2\xi^2 m_3m_4}\frac{T_4}{[T_3]^{\pm 1}}\right]\right.$$
$$\left.+\frac{S_L(m_3^4-\xi^4)}{4\xi^2 m_3m_4}\frac{T_4}{[T_3]^{\pm 1}}\left[\frac{T_1}{T_3}\right]^{\mp 1}+\frac{(\xi^2-m_3^2)^2}{4\alpha\xi^2 m_3}\right\}=-\frac{4\alpha\xi^2 m_3}{(\xi^2-m_3^2)^2} \quad (35)$$

In the absence of varying temperature equation (35) reduces to

$$\left[\frac{T_1}{T_3}\right]^{\pm 1}+\frac{\rho_L\omega^2\alpha(m_3^2+\xi^2)}{\rho\delta^2 m_4(\xi^2-m_3^2)^2}\frac{T_4}{[T_3]^{\pm 1}}+\frac{4\alpha\xi^2 m_3H}{(\xi^2-m_3^2)^2 m_2}\left[\frac{T_1T_2}{T_3}\right]^{\pm 1}\left\{\left[\frac{T_1}{T_3}\right]^{\mp 1}\left[1+\frac{\rho_L\omega^2(m_3^2+\xi^2)}{4\rho\delta^2\xi^2 m_3m_4}\frac{T_4}{[T_3]^{\pm 1}}\right]\right.$$
$$\left.+\frac{(\xi^2-m_3^2)^2}{4\alpha\xi^2 m_3}\right\}=-\frac{4\alpha\xi^2 m_3}{(\xi^2-m_3^2)^2} \quad (36)$$

If $H \to 0$ i.e. for a thermally insulated plate, the equation (36) reduces to

$$\left[\frac{T_1}{T_3}\right]^{\pm 1}+\frac{\rho_L\omega^2\alpha(m_3^2+\xi^2)}{\rho\delta^2 m_4(\xi^2-m_3^2)^2}\frac{T_4}{[T_3]^{\pm 1}}=-\frac{4\alpha\xi^2 m_3}{(\xi^2-m_3^2)^2} \quad (37)$$

If $H \to \infty$ i.e. for a isothermal plate, the equation (36) becomes

$$\left[\frac{T_1}{T_3}\right]^{\mp 1}+\frac{\rho_L\omega^2(m_3^2+\xi^2)}{4\rho\delta^2\xi^2 m_3m_4}\frac{T_4}{[T_1]^{\pm 1}}=-\frac{(\xi^2-m_3^2)^2}{4\alpha\xi^2 m_3} \quad (38)$$

If $\rho_L \to 0$ i.e. in the non-appearance of liquid, equations (37) and (38) respectively reduces to

$$\frac{T_1}{T_3}=\left[-\frac{4\alpha\xi^2 m_3}{(\xi^2-m_3^2)^2}\right]^{\mp 1} \quad (39)$$

The equations (39) and (40) are the same as obtained by Sharma and Pathania [8], Sharma et. al. [7] and in elastokinetics for stress-free boundary conditions conferred in detail by Graff [5].

## 8. SOLUTION OF THE SECULAR EQUATION

The complex transcendental secular equations (28) contain a plethora of information like wave number, phase velocity, attenuation coefficient, etc. To solve these secular equations, we take

$$\frac{T_1}{T_3}=\left[-\frac{(\xi^2-m_3^2)^2}{4\alpha\xi^2 m_3}\right]^{\pm 1} \quad (40)$$

Here $V$ and $Q$ represent phase velocity and attenuation coefficientof plane waves respectively. Also $\xi = R+iQ$, $R$ and $Q$ are real numbers and $R = \frac{\omega}{V}$. The exponential $e^{i\xi(x-ct)}$ in the time-harmonic plane wave solution (18) turns out to be $i(Rx - \omega t) - Qx$ .The various modes of propagating waveforthe attenuation coefficientQ and phase velocity $V$ can be obtained by substitutingequation (41) in dispersion equation (28). Thevalues of the attenuation coefficient $Q$ and phase velocity $V$ are computed by using relation (41).

## 9. NUMERICAL RESULTS AND DISCUSSION

In this section, we present some numerical results for the aluminum-epoxy composite material with the opinion of demonstrating the theoretical results obtained in the previous sections. The numerical values for the material are given as [8,11]:

$$\varepsilon = 0.073,\ \lambda = 7.59\times10^{10}\,Nm^{-2},\ \mu = 1.89\times10^{10}\,Nm^{-2},\ T_0 = 23^0C,\ \rho = 2.19\times10^3\,kgm^{-3},$$
$$K = 2.508\ Km^{-1}/{}^0C,\ C_e = 961.4Jkg^{-1}/{}^0C,\ \bar{\beta} = 280,\ \omega^* = 4.347\times10^{13}s^{-1}$$

For numerical calculations, the fluid taken is water. The density of the fluid is $\rho_L = 1000\ Kg\,m^{-3}$ and the sound velocity in the fluid is $c_L = 1.5\times10^3\,m/\sec$.

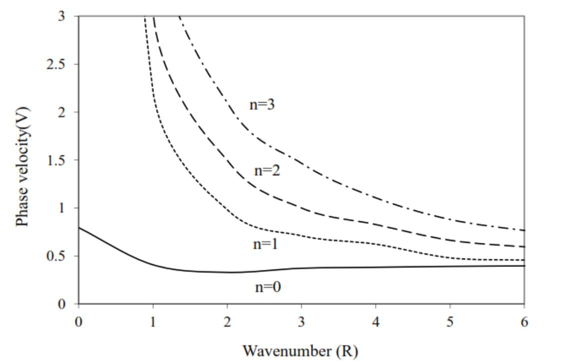
**Table 1: The values of specific heat of water at different temperatures**

| $T_0^*({}^0C)$ | 0 | 10 | 20 | 30 | 40 | 50 | 100 |
|---|---|---|---|---|---|---|---|
| $C_V^*(J/Kg^0C)$ | 1.008 | 1.002 | .9995 | .9987 | .99869 | .9919 | 1.0076 |

Figure 2 and Figure 3 represent the profiles of phase velocity $(V)$ with respect to the wavenumber $(R)$ of symmetric and asymmetric modes of wave propagation with ideal (inviscid) liquid loading respectively. Itis noticed from Figure 2 that for the bottommost symmetric mode $(n=0)$ the phase velocity declines approximately from unity at a long wavelength and with the increasing wave number it moves towards the thermoelastic Rayleigh wave velocity. For other symmetrical modes $(n=1, n=2, n=3)$, the phase velocityaccomplishesmoderately high values at vanishing wavenumber and at short wavelength, it decreases asymptotically and becomes closer to shear wave velocity.

In the case of Figure 3, the value of phase velocity of the lowermostskew-symmetric mode rises from zero at starting wavenumber and thenremains almost constant and approaches to thermoelastic surface wave velocity with the increase in wavenumbers. In higher modes, thepropagating phase velocities have high values at starting wavenumber thatsharply decreases and attains steady and asymptotic Rayleigh wave velocity with advanced wavenumber. For skew-symmetric optical modes the profiles of phase velocity with respect to wavenumber follow

**Figure 2 :** Symmetrical mode




the samemovements as that observed in the case of symmetric one. In both cases, for higher modes the value of phase velocity is found to progress at anamountnearly n-times the phase velocity magnitude of the initial mode.

**Figure 3:** Asymmetrical mode

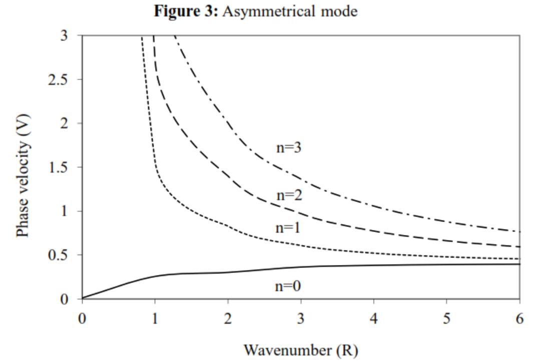
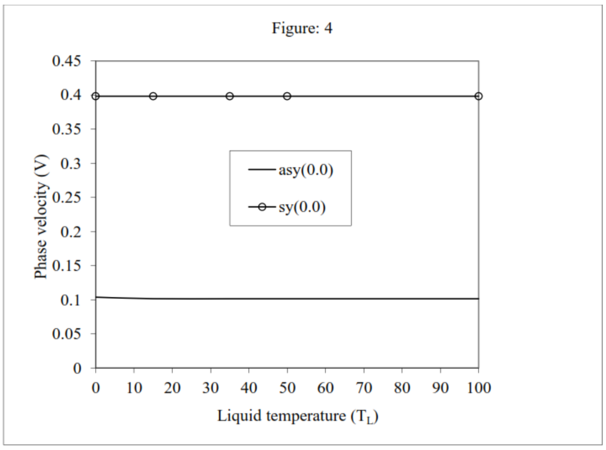
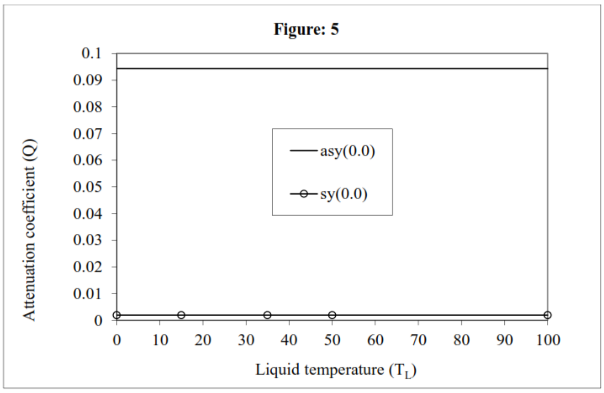



Figure 4 indicates the profiles of phase velocities with respect to liquid loaded temperature changeand it is noticed thatthe propagating phase velocity of non-viscousfluid almost remains constant with fluid loaded temperature change in both symmetric and asymmetric modes except variation in values of the phase velocity and magnitude of symmetrical mode is higher than the magnitude of asymmetric mode. It is also found that the symmetric and asymmetric modes of phase velocity profile have non-dispersive nature with respect to the liquid loaded temperatures i.e. there is no effect of different temperature loading.

Figure 5 indicates the variations of attenuation coefficients with respect to the liquid loaded temperatureof wave propagation for symmetrical and skew-symmetrical modes. Here we noticed that the attenuation coefficient profile with respect to fluid loaded temperature follows the samemovement except for variations in the magnitude of the attenuation coefficient. In this case, the asymmetric mode has a significantly high magnitude than the symmetric one.

In both Figures 4 and 5, it is observed that liquid loading temperature has the opposite effect in symmetric as well as the asymmetric mode in case of phase velocity and attenuation coefficient.




## 10. CONCLUSIONS

The Lamb waves propagation in a thermally conducting elastic homogeneous isotropic plate in the presence of non-viscousfluid layers on its both sides, with varying temperature is studied in the frame of reference of coupled thermo-elasticityand following conclusions are obtained.

1. There exists a coupled system of three types of waves viz. longitudinal waves, the vertical component of transverse waves and waves due to thermal variation in the solid plate.

2. It is found that apart from this coupled system of waves, there is a horizontal component of transverse wavesthat keeps itself isolated from the rest of the coupled motion and is not affectedby the mechanical and thermal load.

3. Apart from the waves in a solid plate, two mechanical waves in each liquid layer are also exists due to mechanical stresses.

4. With the variation in the parameter wavenumber , we find three different regions of secular equations.

5. For asymmetric mode, the plots of the attenuation coefficient with varied liquid temperatures show a decreasingtrend. Thus, it is inferred that the skew-symmetric mode is more sensitive and quite useful in ultrasonic applications.

6. The present analysis is very much useful in the field of earthquake engineering, soil dynamics, seismology, hydrology and geophysics.

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