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Study on Isomorphism of Finite Groups and It's Impact

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***Abstract-** A present paper to analysed the "Study on isomorphism of finite groups and it's impact." The study has shown that G/N satisfy the hypothesis of the theorem. For a group G is finite and the Arbitrary prime is p, we define a characteristic subgroup S-p(G), which is a generalization of the F (Frattini) subgroup of G, as follows: S-p(G) is the intersection of all increasing subgroups M of G such that G has the general degree index of M is G and M are mixed and co-prime to p\ if there is no. such maximal subgroup then we set Sp(G) = G. We obtain some results which characterize solvable groups. Some properties of the subgroup $\sum_p^{\cdot} \mathrm{G}$ =Sp(G) are also obtained. A characteristic subgroup Sp(G) characterize, compare and quantify the effectiveness and complexity of invariants for group isomorphism. The center inner automorphism commutator subgroups and groups, solvable radicals, abelian radicals, derived series and π-radicals and fitting groups. A depends on the type of symmetry Derivative series with basis in lower/upper and central part of factorial series. We see dimension of finite group has one greater and indivisible factor of the direct dimension.*

***Keywords:** Isomorphism, Finite Groups, Maximal subgroup, and consequently solvable group.*

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INTRODUCTION

It is interesting to consider generalization of a finite group with fractional Frattini subgroups and to study the group theoretic properties like solvability, supersolvability, p-solvability, p-supersolvability etc. of the group in the light of the structure of such a subgroup, which are generalisations of Frattini subgroup, and studied their role on solvable, super solvable and nilpotent groups. The study has consider the subgroup S-p{G} and investigate its influence on solvable group. We consider here all finite groups.

Definition- Let M is a maximal sub-group of finitegroup H and G and IT be two normal subgroups of G. The factor group H/K is called a chieffactor of finite G if G do not exist on normal subgroup A of G such that K ⊂ A ⊂ H with proper inclusion H is called a normal supplement of M on G if MH = G. M in G normal index has to defined as the orders of the chieffactor H/K, So the H is minimal on the set of normal-supplements of M-G, and is denoted by n(G : M).

Definition. Let G be any group and p be any prime. Define three characteristic subgroups of G as follows:

Sp(G) = n{M : M €

ϕp(G) = n{M :M.£p(G)}

L(G) = n{M : M € A(G)}

where

$\sum_p^{\cdot}$ Gy~]((j) = {M : M<- G,[G : M]P = 1 and [G : M] is composite}

7P(G) ='{M : M<- G and. \G : M]p = 1}

and

A(G) = {M • M<- G and [G : M] is composite}

If $\sum_p^{\cdot}(\mathrm{G})$ is empty then we define SP(G) = G and the same thing is done for the other two subgroups.

The subgroup Sj>(G)

Definition. Let G be a group and p a prime. Consider the family $\sum_p^{\cdot}(\mathrm{G})$ = {M<- G\n(G : M)p = 1 and r)(G : M) is composite}, where n(G : M)p denotes the p-part of r}(G : M). Define S-p(G) = fl{M : M € $\sum_p^{\cdot}(\mathrm{G})$}. If $\sum_p^{\cdot}(\mathrm{G})$ is empty then we define G = Sp(G).

Remark. We use Italic V in S-p(G) in order to distinguish it from SP(G), which has been defined earlier. We can show by example that SP(G) and S-p(G) are different in general.

Proposition-I: Sp(G) is a characteristic subgroup of G.

Proposition-II: Sp(G) contains Z(G), the center of G and H(G), the hypercentre of G.

Proposition-III: Let 'N' be a normal' sub-group of G.

Then Sp(G)$\frac{N}{N}$ ⊆ Sp$\frac{G}{N}$

Corollary: Let N be a normal subgroup of G. IfN C S-p(G) then Sp(G/N) = Sp(G)/N.

Theorem-2 If G is p-solvable then Sp(G) is solvable.

Proof. Assume that the theorem is false, and let 'G' be a counter aspects of minimal p order. Let' N' be a minimalnormal of sub-group of G exit in Sp(G). By minimal of 'G', we have Sp(G/N) is solvable. Since Sp(G/N) = Sp(G)/N [by Corollary 3.4, So, Sp{G)/N is solvable. Let Nx be another minimal-normal sub-group of 'G' is contained in Sp(G). Then Sp(G)/Nx is solvable. Now since Sp{G) = Sp(G)/N D N\ is isomorphic to a subgroup of the solvable group Sp(G)/N X Sp(G)/Nx, so Sp(G) is solvable and contradiction.

Now we suppose that 'N' is unique minimalnormal, sub-group of 'G', which is contained in Sp(G). Further let B be another minimalnormal, sub-group of 'G'. Then by minimality of G, Sp(G/B) is solvable. Since Sp(G) B/B C Sp(G/B) [by Proposition 3.3], so Sp(G) B/B is solvable.

Now since Sp(G) B/B S Sp(G)/Sp(G)n B = SV(G), so SP(G) is solvable, a contradiction. We may now assume that N is the unique minimal normal subgroup of G. Since G is p-solvable, N is either a p-group or a p'-group. If N is a p-group then N is solvable and hence Sp(G) is solvable, a contradiction. Now 'N' is a p'-group i.e.,\N\P = 1. If SP(G) = G then 'G' is solvable groups and hence Sp(G) is also solvable and contradiction. So it's assume SP(G) ^ G. We now consider two cases.

Case-1. N C SP(G). Since SP(G) is solvable by Theorem 8 [12] N is solvable. Consequently Sp(G) is solvable groups is contradiction.

Case-2. N % SP(G) to exists a maximal-subgroup of 'M' in $\sum_{p}^{\cdot}(G)$ such that N % M. Since N % M so G = MN and hence t/(G : M) = \N\. Again since [G : M] is composite and [G: M] divides r){G : M), so n(G : M) is composite. Also r}(G : M)p = 1 as |JV]P = 1. Hence M E $\sum_{p}^{\cdot}(G)$and consequently N C Sp(G) C M, a contradiction.

Hence the theorem:

In [Theorem 3] it was shown that if 'p' are largestprime divided such group 'G' then SP(G) is solvable. But this result is not true for the subgroup Sp(G). We know that; the alternating group A$ of degree 5 is a non-commutative simple group. Let G = As and p — 5. Now |As| = —1 = 60. And largest prime division |G| is 5. Since G simple, r/(G : M) =]G| for each maximal sub-group G on M. So maximal sub-group does not existing any maximal, sub-group M, G i.e., $\sum_{p}^{\cdot}(G)$(G : M)5 = 1 and consequently Yhp(G) empty. By definition of Sp(G), G = Sp(G). Obviously Sp(G) is not solvable.

Theorem-3 Assume that G is p-solvable.

G is solvable groups: Only if G/Sp(G) is solvable.

Theorem 3.1. If \Sp(G)\3> — 1 then G is super-solvable only if G/Sp(G) is super solvable. Proof. Let 'G' be super solvable. The obviously G/Sp(G) is super solvable. Conversely let G/Sp(G) be super-solvable. We first show that Y^-p(.G) is empty.

If possible: let there exist M in $\sum_{p}^{\cdot}(G)$Then M/Sp(G) is a maximal subgroup of G/Sp(G). So tj(G : M) = r}(G/Sp(G) : M/Sp(G)) = [G/Sp(G) : M/Sp(G)] is a prime, a contradiction. So Yp(G) is empty and consequently G = .Sp(G), We shall now show that A(G) is empty. If posible, let there exist M in A(G). Then [G : M] is composite and so i){G : M) is composite. By hypothesis, \G\r — |5V(G)|P = 1 and consequently r)(G : M)p = 1. So M belongs to ^2-p{G), a contradiction. Hence A(G) is empty and consequently G = L(G).

Since, L(G) is super solvable, G is super solvable. Theorem 3.9. Let p,q be two distinct-primes. Suppose that G is either p-solvable or q-solvable. Then S-p(G) fl Sq(G) is super solvable.

Proof:- Let H denoting on intersection on S-p{G} and Sq(G). By Theorem (3.5), H is solvable. We now assume that H is not super solvable and let 'G' be a counters of minimal- order. If p does not divide \G\ then |G|P = 1. Let M belong to A((?). Then clearly r)(G : M)P = 1 and rj(G : M) is composite.

So Me Ylv(G). Therefore A(G) C ^^(G) and consequently S-p(G) C L(G). By a result in [4], S-p(G) is super solvable and hence H is super solvable, a contradiction. Similarly if q does not divide |G| then H is super solvable. So the eqn. assume that "p and q" are two-distinct primes-dividing |<-?|. Let 'N' be a minimal-normal sub-group of 'G' containing with 'H'. By induction S-p(G/N) n Sq(G/N) is super solvable. Using Corollary (3.4) SV(G/N) n Sq(G/N) = ST(G)/NnSQ(G)/N = H/N. Hence H/N is super solvable. Let Ni be another minima-normal sub-group of G. Then H/Nx is super solvable. So H/N X H/Ni is super solvable.

`Since H = H/N n Ni is isomorphic to a subgroup of H/N x H/N% , so H is super solvable, a contradiction. Its may assume on 'N' are unique minimal-normal sub-group of 'G' contained in 'H'. H being solvable, N is elementary abelian. If |iV| is prime then N is cyclic and hence H is super solvable, a contradiction. So we assume that \N\ is composite. We note that (G) ^ G since (G) is solvable. It is distinguished two and above cases.

Case-1. N C Since both S-p(G) and Sq(G) contain 4>{G) so (f>(G) C H. It follows' that H/(G) = (H/N)/((G)/N). and hence H/(G) is super solvable. By Theorem-9 [12], H is super-solvable, that was contradiction.

Case-2. N £ 4>(G). N £ 4>(G) is exist the maximal sub-group 'M of G' such that' N ^ M. So G = MN' and by {k} definition of normal-index $\sum_{p}^{\cdot}(G)$](G : M) = |AT|.

We claim that |JV|P ^ 1. If possible, let |JV|P = 1. Then M belongs to ^f-p(G) and consequently N C H C Sp(G) C M, a contradiction. Hence N is an elementary abelian p-group. Then $\sum_{p}^{\cdot}(G)$ |JV|g = 1 and consequently M belongs to So N C H C Sq(G) C M, a contradiction. Hence Sp{G) D Sq(G) is super-solvable.

Corollary 3.10. Let p,q is two distinct-primes. Assume that G is either p-solvable or q-solvable. Then (i) if 'Sp{G}=Sq(G)=G,' it follows on G is super solvable, (ii) if S-p(G) = Sq(G) then S-p(G) is super-solvable.

One of the difficulties for a complexity analysis stems from the group isomorphism problem. It is known that, the isomorphism 'problem' of finite groups is found to be one of the few parameters, hence Addition functions of indeterminate complexity in computational group theory. The principle elementary abelian algorithms with an efficient worst case running time measured in the number of generators through which the groups are given. However, we do not even have algorithms with an efficient worst case running time when measured in the order of the group.

Truly Tarzan's Classic 'N' the only improvement for worse cases and complications compared to 'log(n)+O(1)' algorithms are 'nØ1ñc·log'(n)+O(1)' algorithms with a small constant C' depends on the calculation model of computation with randomization, quantum computing etc. There is however a nearly-linear time algorithm that solves group isomorphism for most orders.

Thus, there is a related problem of analyzing 'isomorphism invariants' to decompose groups. The nearest one is easily obtained by adding them together for a general isomorphism test. this Additionally, there are enough complete refractions to be calculated efficiently. However, we don't insufficient how to calculate complete invariants 'efficiently' special cases, such as nil potent p-groups of class II.

Partial invariants homogeneity of changes in partial transformation does not give a test, It is kept in the algorithm. It is primarily abundance based algorithmic grouping principles that allow sorting from an older historical perspective. Subset of uniqueness of theories in groups from the classic. As outlined in these include exploiting the Frattini sub-group Φ(G) [3], the exponent-p-central series, characteristic series and similar.

Overall, many of the techniques currently in use are ad-hoc, focused on practical performance ND do not lead to efficient worst case upper bounds for the complexity of the algorithmic problems.

As a consequence, the general picture for finite groups is somewhat chaotic. There is often non-structured path with comparing invariants for one group Isomorphism. To given invariants on incomparable to their distinguishing-power and or makes over unclear which invariant to use. Also, the required time to evaluate an invariant may be difficult to estimate and can depend significantly on the input group. Even when we are given a class of efficiently computable invariants, it will generally be unclear which invariants to choose or how to efficiently combine their evaluation algorithmically.

4. Some solvability and supersolvability conditions-

Theorem- Let G be p-solvable and Yj-piP)^ . Then G is solvable if and only if r}{G : M) = [G:M] for each M in Y^v(G)-

Proof- Let G be solvable.

Than by Corollary of Theorem-1 [2], we have $\sum_{p}^{\cdot}(G)$: M) = [G : M] for each M in Yhp{G). Conversely, let r/(G : M) = [G : M] for each M in We shall show that G is solvable. If possible, let G be not solvable and a counter example of minimal order. Let G be simple, and M belong to J2V(G).

Then by hypothesis |G| = r)(G : M) = [G : M], This implies that M =< 1 > and so G is cyclic and hence solvable, a contradiction. So G is not simple. Let N be a minimal-normal sub-group of G. It can be shown easily that G/N satisfies the hypothesis of the theorem. By minimality of G, G/N is solvable. Let N\ be another minimal-normal sub-group of G. Then G/N\ is solvable. So G/N x G/N\ is solvable. Since G = G/Nfl Ni is isomorphic to a subgroup of G/N x G/Ni so G is solvable, a contradiction. So we assume that N is the unique minimal-normal snb-group of G $\sum_{p}^{\cdot}(G)$. We note that Sp(G) G, otherwise G will be solvable, which is a contradiction. We now consider two cases. Case-1. N C Sp(G). Then JV is solvable because Sp(G) is solvable by Theorem (3.5). As G/N is solvable so G is solvable, a contradiction.

Let p,q be two distinct primes. Assume that G is either p-solvable or q-solvable. Then

(i) if S-p(G) = Sq(G) = G, it follows that G is supersolvable,

(ii) if S-p(G) = Sq(G) then S-p(G) is supersolvable.

Some solvability and supersolvability conitions:

Theorem 4- Let G be p-solvable and Yj-piP) ^ . Then G is solvable if and only if r}{G : M) = [G : M] for each M in $\sum_p^{\cdot}(G)$.

Proof. Let G be solvable. Than by Corollary of Theorem-1 [2], we have $\sum_p^{\cdot}(G)$ (G : M) = [G : M] for each M in Yhp{G}. Conversely, let r/(G : M) = [G : M] for each M in We shall show that 'G' is solvable. If possible, let 'G' be not solvable and a counter example of minimal order. Let 'G' be simple, and M belong to J2V(G). Then by hypothesis |G| = r)(G : M) = [G : M], This implies that M =< 1 > and so 'G' is cyclic and hence solvable, a contradiction. So G is not simple. Let 'N' be a minima- normal sub-group of 'G'. It can be shown easily that G/N satisfies the hypothesis of the theorem. By minimality of 'G', G/N is solvable. Let N\¥ be another minimal-normal sub-group of G. Then G/N\ is solvable.

So G/N x G/N\ is solvable. Since G = G/Nfl Ni is isomorphic to a subgroup of 'G/N x G/Ni' so G is solvable, a contradiction. We assume 'N' is the unique minimal-normal sub-groups of G. We note that Sp(G) G, otherwise G will be solvable, which is a contradiction. We now consider two cases.

Case-1. N C Sp(G). Then JV is solvable because Sp(G) is solvable. As G/N is solvable so G is solvable, a contradiction.

So there N % (G). Then there exists a maximal subgroup Mi of G such that G -MiA. Let q be a prime divisor of [G : Mi]. We claim that A C q(G). If possible, let A % 4>q{G). Then there exists a maximal sub-group Mi in 7q(G) such that G = M2A. By definition of normal index, t)(G : Mi) = |A| — r){G : M2}. Thus 'M1 and M2' are non-normal maximal sub-groups belonging to: of equal normal index. So by hypothesis 'M1 and M2' are conjugate in 'G'. Hence |Mi| = |M2| and consequently [G :Mi] = [G:M2], which is impossible, because q divides [£?: Mi] but [G : A^], = 1. So, A C 4>q(G) and hence A is solvable because g(G) is solvable. Consequently, G±£ is solvable, a contradiction. This completes the proof.

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