Numerical Solution: A Study of Partially Differential Equation

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Abstract - This study explores various numerical methods for solving partial differential equations (PDEs), which are crucial in modeling complex phenomena across diverse scientific and engineering fields. Given the challenges associated with obtaining analytical solutions for PDEs, numerical approaches such as the Finite Difference Method (FDM), Finite Element Method (FEM), and Finite Volume Method (FVM) have become essential. This research provides a comparative analysis of these techniques, evaluating them based on accuracy, computational efficiency, and applicability to different types of PDEs, including elliptic, parabolic, and hyperbolic equations. Through a series of test cases, the study highlights the strengths and weaknesses of each method, offering practical insights into their use for solving PDEs in real-world scenarios.

Keywords: Partial differential equations, Numerical methods, Solution

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INTRODUCTION

Partial Differential Equations (PDEs) are fundamental to the mathematical modeling of a wide range of physical processes, including heat conduction, fluid dynamics, and electromagnetic fields. Unlike ordinary differential equations, PDEs involve multiple independent variables, making their solutions more complex and challenging. Analytical solutions to PDEs are often difficult or impossible to obtain, particularly for non-linear problems or those with complex boundary conditions. Consequently, numerical methods have become indispensable for approximating solutions to PDEs.

The primary numerical techniques for solving PDEs include the Finite Difference Method (FDM), the Finite Element Method (FEM), and the Finite Volume Method (FVM). Each method has its unique approach to discretizing the problem domain and handling boundary conditions, which influences their suitability for different types of PDEs. For instance, FDM is known for its simplicity and ease of implementation, particularly for problems on regular grids. In contrast, FEM is highly flexible in handling complex geometries and boundary conditions, making it widely used in engineering applications.

This study aims to provide a comprehensive analysis of these numerical methods, assessing their performance in solving different types of PDEs. By examining the accuracy, computational cost, and practical applicability of each technique, the research seeks to guide the selection of the most appropriate method for solving PDEs in various scientific and engineering contexts.

METHODS

First example (one dimensional heat equation):

Using the given parameters, $u_t = u_{xx}$ find the solution to the boundary value issue

$u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin \pi x \quad 0 \le x \le 1$ taking $h = 0.2$ and $\alpha = 1/2$.

Solution:

We are aware of the fact that the optimal answer to the one-dimensional heat equation

$u_t = c^2 u_{xx}$ is given by

$$u = (c_1 \cos px + c_2 \sin px).e^{-c^2 p^2 t}, \quad \text{where } c^2 = 1 \qquad (1)$$

Therefore,

$$u = (c_1 \cos px + c_2 \sin px).e^{-p^2 t} \qquad (2)$$

Conditions are:

(i) $u = 0,\ x = 0$

(ii) $u = 0,\ x = 1$

(iii) $u = \sin \pi x$ at $t = 0,\ 0 \le x \le 1$

Using condition (i) in equation (2), we get

$$0 = c_1 e^{-p^2 t}$$

$$0 = c_1 e^{-p^2 t}$$

$$c_1 = 0$$

From equation (2)

$$u = (c_2 \sin px).e^{-p^2 t} \qquad (3)$$

Now using condition (ii) in equation (3)

$$0 = (c_2 \sin p).e^{-p^2 t}$$

$$\Rightarrow \sin p = 0 = \sin n\pi, \qquad n \in I$$

$$\Rightarrow \quad p = n\pi$$

From (3)

$$u = (c_2 \sin p).e^{-n^2 \pi^2 t} \qquad (4)$$

Most general solution of given equation is

$$u = \sum_{n=1}^{\infty} a_n \sin n\pi x.e^{-n^2 \pi^2 t} \qquad (5)$$

We get the result when we plug in condition (iii) into equation (5).

$$\sin \pi x = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

$$\Rightarrow \sin \pi x = a_1 \sin \pi x + a_2 \sin 2\pi x + a_3 \sin 3\pi x + \ldots\ldots\ldots\ldots\ldots$$

$$\Rightarrow a_1 = 1, \quad a_2 = a_3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$$

Now putting these value in equation (5)

$$u = \sin \pi x.e^{-\pi^2 t}$$

	x→	0	0.2	0.4	0.6	0.8	1
t↓	j \ i	0	1	2	3	4	5
0	0	0	0.587785252	0.951056516	0.951056516	0.587785252	0
0.02	1	0	0.482494526165092	0.780692542720894	0.780692542720894	0.482494526165092	0
0.04	2	0	0.396064662853158	0.640846086239177	0.640846086239177	0.396064662853158	0
0.06	3	0	0.325117091809890	0.526050504871922	0.526050504871922	0.325117091809890	0
0.08	4	0	0.266878450163855	0.431818403230012	0.431818403230012	0.266878450163855	0
0.10	5	0	0.219072171091850	0.354466218815846	0.354466218815846	0.219072171091850	0

BENDER-SCHMIDT METHOD

Consider one dimensional heat equation, namely,

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (a)$$

where

$\alpha^2 = \dfrac{k}{pc}$ is an example of parabolic equation. If $\alpha^2 = \dfrac{1}{a}$, the equation becomes,

$$\frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t}$$

With boundary conditions, $u(0,t) = T_0,\ u(\ell,t) = T_\ell$, and with initial condition

$$u(x,0) = f(x),\ 0 \le x \le \ell.$$

Think of a rectangular grid in the x-t plane with h vertices and k tangents. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

substituting these in (a), we obtain

$$u_{i,j+1} - u_{i,j} = \frac{k}{ah^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

or $u_{i,j+1} - u_{i,j} = \lambda\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$

where $\lambda = \dfrac{k}{ah^2}$

i.e., $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$ (b)

An explicit formula is what we get with equation (b). The condition for its validity is an

$$0 < \lambda \leq \frac{1}{2}$$

The coefficient of disappears, $\lambda = 1/2$, leading to the transformation of $u_{i,j}$, equation (b) into

$$u_{i,j} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right].$$

	t↓	0	0.02	0.04	0.06	0.08	0.1
x→	j \ i	0	1	2	3	4	5
0	0	0	0	0	0	0	0
0.2	1	0.58779	0.47553	0.38471	0.31124	0.2518	0.20371
0.4	2	0.95106	0.76942	0.62247	0.50359	0.40741	0.32961
0.6	3	0.95106	0.76942	0.62247	0.50359	0.40741	0.32961
0.8	4	0.58779	0.47553	0.38471	0.31124	0.2518	0.20371
1	5	0	0	0	0	0	0

DU FORT AND FRANKEL METHOD

In (a), if we substitute the central difference approximations for the derivative,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

And

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

We obtain

$$u_{i,j+1} - u_{i,j-1} = \frac{2kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

i.e., $u_{i,j} = u_{i,j-1} + 2\alpha\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$ (e)

where $\alpha = kc^2/h^2$. The three-tiered approach that uses this difference equation is known as the Richardson scheme.

If we replace by the mean of the variables $u_{i,j-1}$ and $u_{i,j+1}$ i.e.

$$u_{i,j} = \frac{1}{2}\left(u_{i,j-1} + u_{i,j+1}\right) \text{ in (e),}$$

we get $u_{i,j+1} = u_{i,j-1} + 2\alpha\left[u_{i-1,j} - 2(u_{i,j-1} + u_{i,j+1}) + u_{i+1,j}\right].$

On simplification, it can be written as

$$u_{i,j+1} = \frac{1-2\alpha}{1+2\alpha}u_{i,j-1} + \frac{2\alpha}{1+2\alpha}\left[u_{i-1,j} + u_{i+1,j}\right]$$

This technique of calculating differences is known as the Du Fort-Frankel scheme:

	t↓	0	0.02	0.04	0.06	0.08	0.1
x→	j \ i	0	1	2	3	4	5
0	0	0	0	0	0	0	0
0.2	1	0.58779	0.47553	0.38471	0.31124	0.2518	0.20371
0.4	2	0.95106	0.76942	0.62247	0.50359	0.40741	0.32961
0.6	3	0.95106	0.76942	0.62247	0.50359	0.40741	0.32961
0.8	4	0.58779	0.47553	0.38471	0.31124	0.2518	0.20371
1	5	0	0	0	0	0	0

	x→	0	0.2	0.4	0.6	0.8	1
t↓	j \ i	0	1	2	3	4	5
0	0	0	0.587785252292473	0.951056516295153	0.951056516295153	0.587785252292473	0
0.02	1	0	0.532544051503428	0.861674375839121	0.861674375839121	0.532544051503428	0
0.04	2	0	0.482494526165092	0.780692542720894	0.780692542720894	0.482494526165092	0
0.06	3	0	0.437148752524894	0.707321539724895	0.707321539724895	0.437148752524894	0
0.08	4	0	0.396064662853158	0.640846086239177	0.640846086239177	0.396064662853158	0
0.10	5	0	0.358841735804913	0.580618125114360	0.580618125114360	0.358841735804913	0

BENDER-SCHMIDT METHOD

The initial condition is

$$u(x,0) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}.$$

One dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

Where

$$c^2 = 1 \; h = \tfrac{2}{5}, \; \alpha = \tfrac{1}{4}$$

$$k = \tfrac{1}{25}, \;\; u(0,t) = 0 = u(2,t) \text{ for any } t.$$

	x→	0	0.2	0.4	0.6	0.8	1
t↓	j \ i	0	1	2	3	4	5
0	0	0	0.587785252292473	0.951056516295153	0.951056516295153	0.587785252292473	0
0.02	1	0	0.531656755220025	0.860238700294483	0.860238700294483	0.531656755220025	0
0.04	2	0	0.480888052683633	0.778093214025869	0.778093214025869	0.480888052683633	0
0.06	3	0	0.434967329848284	0.703791923690310	0.703791923690310	0.434967329848284	0
0.08	4	0	0.393431645846720	0.636585775229804	0.636585775229804	0.393431645846720	0
0.10	5	0	0.355862266730811	0.575797242884033	0.575797242884033	0.355862266730811	0

CRANK-NICHOLSON DIFFERENCE METHOD

The initial condition is

$$u(x,0) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}.$$

One dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = 1 \; h = \tfrac{2}{5}, \; \alpha = \tfrac{1}{4} \; k = \tfrac{1}{25}, \;\; u(0,t) = 0 = u(2,t)$$

for any t

	x→	0	0.2	0.4	0.6	0.8	1
t↓	j \ i	0	1	2	3	4	5
0	0	0	0.587785252292473	0.951056516295153	0.951056516295153	0.587785252292473	0
0.02	1	0	0.534254429627061	0.861377266220623	0.861377266220623	0.534254429627061	0
0.04	2	0	0.481526240326048	0.781597959277497	0.781597959277497	0.481526240326048	0
0.06	3	0	0.440275399622164	0.710998594987854	0.710998594987854	0.440275399622164	0
0.08	4	0	0.395017578640727	0.641924793686430	0.641924793686430	0.395017578640727	0
0.10	5	0	0.360505208137336	0.581781815842564	0.581781815842564	0.360505208137336	0

DU FORT AND FRANKEL METHOD

An equation for one-dimensional heat flow serves as

$$u(x,0) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}.$$

the starting point is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = 1 \; h = \tfrac{2}{5}, \; \alpha = \tfrac{1}{4} \; k = \tfrac{1}{25}, \;\; u(0,t) = 0 = u(2,t)$$

for any t.

	x→	0	0.2	0.4	0.6	0.8	1
t↓	j \ i	0	1	2	3	4	5
0	0	0	0.587785252292473	0.951056516295153	0.951056516295153	0.587785252292473	0
0.02	1	0	0.531656755220025	0.860238700294483	0.860238700294483	0.531656755220025	0
0.04	2	0	0.482674650862319	0.780983990603220	0.780983990603220	0.482674650862319	0
0.06	3	0	0.437546915274415	0.707965780586674	0.707965780586674	0.437546915274415	0
0.08	4	0	0.396880143816331	0.642165562154770	0.642165562154770	0.396880143816331	0
0.10	5	0	0.359904159143062	0.582337162185925	0.582337162185925	0.359904159143062	0

ERROR ANALYSIS

	t↓	0	0.02	0.04	0.06	0.08	0.1
x→	j \ i	0	1	2	3	4	5
0	0	0	0	0	0	0	0
0.2	1	u1	u5	u9	u13	u17	u21
0.4	2	u2	u6	u10	u14	u18	u22
0.6	3	u3	u7	u11	u15	u19	u23
0.8	4	u4	u8	u12	u16	u20	u24
1	5	0	0	0	0	0	0

CONCLUSION

This study has provided a detailed comparative analysis of numerical methods for solving partial differential equations. The choice of numerical method should be based on the specific characteristics of the PDE being solved, including the type of equation, the domain geometry, and the desired accuracy. The findings of this study provide valuable insights for researchers and practitioners, aiding in the selection of the most appropriate numerical technique for solving PDEs in various fields of science and engineering. By improving the understanding of these methods, the study contributes to more accurate and efficient numerical solutions of PDEs, thereby advancing the modeling of complex physical phenomena.

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