

# A Study on Different Methods to Numerical Solution for Ordinary Differential Equations

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**Abstract** - Different numerical approaches to solving ordinary differential equations (ODEs) are compared in this research. It is critical to have a reliable and effective solution for ODEs because of its widespread use in modelling dynamic systems in many branches of science and engineering. Numerous popular numerical methods, such as the Finite Difference method, the Runge-Kutta methods, and Euler's method, are assessed in this study. We evaluate these strategies' efficacy according to their precision, computational efficiency, and practicality. This article highlights the merits and limits of each technique via a series of benchmark problems, providing insights into their application to various forms of ODEs. In order to improve the accuracy and consistency of ODE solutions, the results should help in choosing the right numerical approaches for real-world applications.

**Keywords:** Different methods, Numerical solution, ordinary differential equation

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## INTRODUCTION

A wide variety of systems in engineering, biology, and physics may be mathematically represented using ODEs. Their description of the function-derivative connection captures the dynamics of systems in which the state determines the nature of the change. Although it is ideal to find analytical solutions to ODEs, this isn't always possible, particularly with more complicated or non-linear equations. Consequently, numerical approaches are now essential for estimating solutions to ordinary differential equations.

In order to solve an ODE computationally, numerical techniques reduce the issue to a system of algebraic equations. The Finite Difference method, Euler's method, and the Runge-Kutta methods are among the most popular approaches. diverse issues and circumstances call for diverse approaches, as each strikes a unique balance between accuracy, computing cost, and implementation simplicity.

The purpose of this research is to compare these approaches in depth, taking a close look at their advantages and disadvantages as well as the kinds of ODEs for which they are most suited. As a result, the study hopes to provide useful recommendations for choosing the best numerical approach to solving certain kinds of ordinary differential equations.

## SIMILARITY TRANSFORMATION

Finding solutions to balanced problems is easy. Both the circle and the symmetrical triangle are naturally imaginable, which makes their symmetry all the more

striking. Just as geometrical evenness helps to organise geometrical issues, a more complicated and real mathematical balance between ordinary and partial differential equations might ease the structuring of the equations. If subjecting anything to a certain test produces the same result, then we say that it is even according to scientific standards. If we rotate the centroid of a symmetrical triangle by 120 degrees, it will still look the same. The length of the triangle is equal because, as we have seen, it does not change when rotated 120 degrees about its centroid. Ordinal and partial differential equations may be invariant under certain sets of logarithmic modifications; this arithmetical invariance, like the geometric invariances, provides a check, as was mentioned above. For the purpose of solving ordinary differential equations, mathematician Sophus Lie first proposed logarithmic balancing in 1875.

## DIFFERENTIAL EQUATIONS TYPES

A number of shapes are shown via differential equations. These differential equations may serve as a roadmap for choosing a solution approach, in addition to describing the equation's features. The most common distinctions are ordinary and partial, linear and non-linear, and homogeneous and non-homogeneous. This is certainly not an exhaustive list; other subclasses and characteristics of differential equations may out to be quite useful in certain contexts.

## Ordinary differential equations

Any system of equations including a real or complex variable  $x$ , as well as an unknown function, its derivatives, and specific functions of  $x$ , is called an ODE. The unknown function is usually represented by a variable ( $y$ ) that  $x$  is dependent on. Because of this,  $x$  is the independent variable in the equation. The adjective "ordinary" is used to differentiate it from partial differential equations, which may have several independent variables.

A linear differential equation is produced when both the unknown function and its derivatives are linear. They have reached a point in their theory where integrals may stand in for solutions in many cases.

As most ODEs found in physics are linear, it is feasible to define most special functions as solutions of these equations.

Since differential equations seldom contain closed-form formulations, numerical methods are often used when calculating computational solutions to these sorts of problems.

## Partial differential equations

A partial differential equation (PDE) consists of an unknown multivariable function and its partial derivatives. (On the other hand, ordinary differential equations deal with derivatives of functions linked to only one variable.) PDEs are useful tools for solving problems involving functions of many variables. Constructing a computer model that solves the issue is another possibility.

Among the many natural phenomena that may be described using PDEs are sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, and quantum mechanics. It is feasible to formalise these seemingly independent physical processes using PDEs in a comparable way. Ordinary differential equations work well for one-dimensional systems while partial differential equations work better for multidimensional systems when describing dynamical systems. When it comes to stochastic PDEs, the generalisation of PDEs is evident.

## Non-linear differential equations

For unknown functions with degrees greater than 1, it is possible to take derivatives; nevertheless, solving these equations as a product of their derivatives produces non-linear differential equations. There aren't many methods for solving nonlinear differential equations accurately, and the ones that do often need certain symmetries in the original equation. Nonlinear differential equations, when examined over extended time intervals, may exhibit complex, chaotic behaviour. Specifically, resolving the well-posedness of the initial and boundary value problems for nonlinear PDEs is considered a significant achievement in mathematical theory, as are the

existence, uniqueness, and extendibility of solutions to these equations (refer to the existence and smoothness of Navier-Stokes equations for comparison). If the differential equation correctly depicts a substantial physical process, a solution to the equation should be expected.

## Homogeneous and non-homogeneous differential equations

A linear ordinary differential equation of order  $n$  is said to be homogeneous if it is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = 0,$$

where  $y' = dy/dx$  i.e., if all the terms are proportional to a derivative of  $y$  (or  $y$  itself) and there is no term that contains a function of  $x$  alone.

A first-order ordinary differential equation might potentially signify something quite different. When expressed in this way, we say that the equation is homogeneous.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Such equations can be solved in closed form by the change of variables  $u = y/x$  which transforms the equation into the separable equation

$$\frac{dx}{x} = \frac{du}{f(u) - u}$$

The non-homogeneous differential equation of this type has the form

$$y'' + py' + qy = f(x)$$

where  $p$  and  $q$  are constant numbers (that can be treated as real or as complex numbers). For each equation we can write the related homogeneous equation.

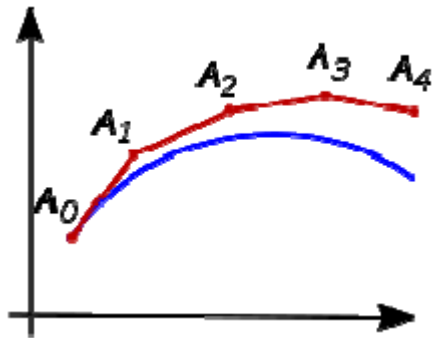
$$y'' + py' + qy = 0$$

## SOME IMPORTANT METHODS TO NUMERICAL SOLUTION FOR ORDINARY DIFFERENTIAL EQUATIONS

### Euler's method

An algorithm called Euler's technique may be used numerically to solve first-order, first-degree differential equations given a starting value. It is the most basic explicit method for numerically integrating ordinary differential equations and the simplest Runge-Kutta technique.

Since it is a first-order method, the Euler algorithm shows that the global error (error at any given moment) and the local error (error per step) are proportionate to each other. The Euler technique is often used as a basis for more complex procedures like the predictor-corrector method.



You need to find the shape of a curve that starts at a given point, and you have a differential equation that explains it. Here, a differential equation is analogous to a formula that, given a point on a curve, calculates the slope of the tangent line to that curve.

Look at the picture up there? The basic idea is that even if we don't know where the curve starts (A0), we do know where it ends (A1). The slope and tangent line of the curve at A0 may be found using the differential equation.

Proceed down the tangent line to get closer to point A1. A1 will be close to the curve since the slope is somewhat constant during this little step. By acting as if A1 is still on the curve, we may apply the same reasoning to it as we did to point A0 before. Multiple steps are required to compute a polygonal curve using the formula A0 A1 A2 A3 A4.

### Improved Euler's Method

The improved Euler's method gives greater improvement in accuracy over the original Euler's method. Here the core idea is that we use a line

through  $(x_0, y_0)$  whose slope is the average of the slopes

at  $(x_0, y_0)$  and  $(x_0, y_1)$  where

$$y_1^{(1)} = y_0 + hf(x_0, y_0)$$

The curve in the interval  $(x_0, x_1)$  is roughly represented by this line. A more widely applicable version of Euler's enhanced formula is

$$y_1^{(n+1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(n)})]; \quad n = 0, 1, 2, \dots$$

where  $y_1^{(n)}$  is the nth approximation to  $y_1$ .

### Modified Euler Method

In this method the curve in the interval  $(x_0, x_1)$  where  $x_1 = x_0 + h$  is approximated by the line through  $(x_0, y_0)$  with the slope  $f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$  which is the slope at the middle point whose abscissa is the average of  $x_0$  and  $x_1$ . A generalized form of Euler's improved formula is

$$y_{n+1} = y_n + hf\left\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right\}$$

### Runge-Kutta Method

In numerical analysis, the Runge-Kutta methods are a family of implicit and explicit iterative methods, which include the well-known routine called Euler method, used in temporal discretization for the approximate solutions of ordinary differential equations. These methods were developed around 1900 by the German mathematicians Carl Runge and Martin Kutta.

As we have shown, the approach proposed by Euler yields

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0'$$

Expanding by Taylor's series

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \dots$$

It follows that the Euler's method agrees with the Taylor's series solution up to the term in  $h$ .

Hence, Euler's method is the Runge-kutta method of the first order.

(ii) Second order Runge-Kutta method:

The modified Euler's method gives

$$y_1 = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_0 + h, y_1)]$$

Substituting  $y_1 = y_0 + hf(x_0, y_0)$  from (1), we derive as follows:

$$y_1 = y_0 + \frac{h}{2} [f_0 + f(x_0 + h, y_0 + hf_0)] \quad \text{where } f_0 = f(x_0, y_0)$$

Expanding L.H.S. by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots \quad \dots(3)$$

Expanding  $f(x_0 + h, y_0 + hf_0)$  by Taylor's series for a function of two variables, gives

$$y_1 = y_0 + \frac{h}{2} \left[ f_0 + \left\{ f(x_0, y_0) + h \left( \frac{\partial f}{\partial x} \right)_0 + hf \left( \frac{\partial f}{\partial y} \right)_0 + O(h^2) \right\} \right]$$

$$y_1 = y_0 + \frac{1}{2} \left[ hf_0 + hf_0 + h^2 \left\{ \left( \frac{\partial f}{\partial x} \right)_0 + hf \left( \frac{\partial f}{\partial y} \right)_0 \right\} + O(h^3) \right]$$

$$y_1 = y_0 + hf_0 + \frac{h^2}{2!} f''_0 + O(h^3)$$

$$\left[ \because \frac{df(x, y)}{dx} = \frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right]$$

$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + O(h^3) \quad \dots\dots\dots(4)$$

Comparing (3) and (4), it follow that the modified Euler's method agrees with the Taylor's series solution upto the term in  $h^2$ .

Hence the modified Euler's method is the Runge-Kutta method of the second order.

∴ The second order Runge-Kutta method formula is

$$y_1 = y_0 + \frac{1}{2} (k_1 + k_2)$$

$$\text{Where } k_1 = hf(x_0, y_0) \quad \text{and } k_2 = hf(x_0 + h, y_0 + k_1)$$

(iii) Third order Runge-Kutta method:

Similarly, it can be seen that Runge's method agrees with Taylor's series solution upto the term in  $h^2$ .

As such, Runge's method is the Runge-Kutta method of the third order.

The third order Runge-Kutta method is

$$y_1 = y_0 + (k_1 + 4k_2 + k_3)/6 \quad n = 0, 1, 2, \dots\dots\dots$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$$

$$k_3 = hf(x_0 + h, y_0 + k') \quad \text{where } k' = hf(x_0 + h, y_0 + k_1)$$

(iv) Fourth order Runge-Kutta method:

This method is most commonly used and is often referred to as Runge-Kutta method only.

$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4)/6 \quad n = 0, 1, 2, \dots\dots\dots$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$$

$$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

(v) Fifth order Runge-Kutta method:

Nikolaos Christodoulou, 2009 has presented the fifth order Runge-Kutta method to solve a first order initial value problem of the

form  $\frac{dy}{dt} = f(x, y)$ ,  $y(x_0) = y_0$ , which is given by the following equation:

$$y_{n+1} = y_n + \frac{7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6}{90} \quad \dots(6)$$

$$\text{where } k_1 = hf(x_n, y_n), \quad k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{4}, y_n + \frac{3k_1 + k_2}{16}\right)$$

$$k_4 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_3}{2}\right)$$

$$k_5 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{-3k_2 + 6k_3 + 9k_4}{16}\right)$$

$$k_6 = hf\left(x_n + h, y_n + \frac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7}\right)$$

## Adams-Bashforth methods

Using the two-step approach as an additional tool, the drawbacks of the one-step strategy in terms of error estimations are addressed. There is an alternative method called Corrector Predictor, however it is unable to start its own launch. Therefore, this technique requires the value of the previous one to start the prediction. The Adams-Bashforth approach is more efficient than one-step methods such as the Runge-Kutta method, which is another benefit. In comparison to the Adams-Bashforth approach, which simply repeats computations based on prior points of interest, the Runge-Kutta method is more time-consuming since it evaluates the  $f(x, y)$  function.

In order to calculate, this fourth-order multistep approach requires four values. You may get these values  $(x_{i-3}, y_{i-3})$ ,



$(x_{i-2}, y_{i-2})$ ,  $(x_{i-1}, y_{i-1})$  and  $(x_i, y_i)$  the initial values

of this approach  $y_{i+1}$  using any one-step approach, Now think about a differential equation.

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad \text{initial conditions} \quad x = x_0, y(x_0) = y_0.$$

$$y_{i+1}^p = y_i + \frac{h}{24} (-9f_{i-3} + 37f_{i-2} - 59f_{i-1} + 55f_i)$$

The Adams-Bashforth corrector formula is the name given to this equation.

$$y_{i+1}^c = y_i + \frac{h}{24} (f_{i-2} - 5f_{i-1} + 19f_i + 9f_{i+1}) \text{ where } f_{i+1} = f(x_{i+1}, y_{i+1}^p)$$

The Adams-Bashforth corrector formula is the name given to this equation.

### Shooting Method

One approach to addressing boundary value problems in numerical analysis is the shooting technique, which involves reducing the issue to an initial value problem system. To put it simply, we "shoot" off trajectories in all sorts of directions until we locate one with the boundary value we want. This shooting technique example may help to understand the subsequent explanation.

This approach is described for a second-order ordinary differential equation boundary value issue.

Let

$$y''(t) = f(t, y(t), y'(t)), \quad y(t_0) = y_0, \quad y(t_1) = y_1$$

serve as the issue with border values. The answer of the initial value issue is represented as  $y(t, a)$ .

$$y''(t) = f(t, y(t), y'(t)), \quad y(t_0) = y_0, \quad y(t_1) = a$$

Define the function  $F(a)$  as the difference between  $y(t_1, a)$  and the specified boundary value  $y_1$ .

$$F(a) = y(t_1, a) - y_1$$

If  $F$  has a root  $a$ , then the solution to the boundary value problem,  $y(t, a)$ , is also a solution to the associated beginning value problem. Conversely, if  $y(t)$  is a boundary value problem solution, then it is also the unique beginning value problem solution  $a = y'(t_0)$ , where  $a$  is a root of  $F$ .

### Finite Difference Method

This approach uses finite difference approximation to solve differential equations instead of derivatives. In particular, the variables  $y'$  and  $y''$  in the equation

$$P \frac{d^2 y}{dx^2} + Q \frac{dy}{dx} + Ry = S \quad (1)$$

are replaced by the right-hand sides of equations

$$y' = \frac{1}{2h} (y_{i+1} - y_{i-1})$$

and

$$y'' = \frac{1}{h^2} (y_{i-1} - 2y_i + y_{i+1})$$

in that order.

### CONCLUSION

This research has compared and contrasted several numerical approaches to solving ordinary differential equations, highlighting the strengths and weaknesses of each. For stiff or very non-linear equations in particular, Euler's approach, while simple and easy to use, often fails to provide adequate results. Many practical applications like the Runge-Kutta algorithms, especially the fourth-order variation, since they provide a favourable trade-off between processing effort and accuracy. Despite its strength, the Finite Difference approach to boundary value problems need cautious discretisation to sidestep stability concerns.

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