https://doi.org/10.29070/yp9tjf09

Kummer and Dixon Summation Theorems: Applications in Hypergeometric Functions and Double Series

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Abstract - The Kummer and Dixon summation theorems essentially carry out the essential task of simplifying hypergeometric functions and double series. This essay will cover both the topic and the basic concept of hypergeometric functions as well as the understanding and importance of summation theorems in mathematical analysis. We explore the theorems made by Kummer and Dixon, show how their theorems are relevant and explain them by using particular examples involving the hypergeometric series. The other topic considered also is regarding the convergence of double series and we explain how these summation theorems can be used to make evaluation of them simpler. The comparative study exhibits the interworking nature between Kummer and Dixon telegrams to describe instances where their compounded application is advantageous. The lecture goes further in the topic and scrutinizes the application of these functions in multivariable hypergeometric as well as mathematics and physics. The method is also discussed. Besides this, the latest headway and unsolved issues are also highlighted with some prospective research targets being listed for the future. The Noether's Theorem illuminates the concise relation between the affinity of the Dixon and Bernoulli Summation Theorems to maths and physics, therefore dispelling any misconceptions about its practicality. Our research aim is to provide the necessary spark for the discovery and invention in this area by developing new approaches and methods which will eventually lead to the growth of mathematical research and its practical applications.

Keywords: Kummer theorem, Dixon theorem, Hypergeometric functions, Double series, Summation theorems, Convergence analysis, Multivariable extensions, Computational techniques, Symbolic computation

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INTRODUCTION

Hypergeometric functions are a special function class defined by their Taylor series expansion. Often used in mathematics and physics. They are generalizations of many known functions and therefore can be used to solve differential equations, estimate integrals and create a series.

1. Superposition of Hypergeometric Functions

Hypergeometric functions is a special function class and exactly this class of functions is determined by the series expansion and it is the generalization of the geometric series. They are solutions for the hypergeometric differential equations and appear consistently nearly every branch of mathematics and physics. The general form of the equation of a hypergeometric function is given by the hypergeometric series ${}_2F_1(a,b;c;z)$, which converges for $|z| < 1$ and is defined as:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

where $(a)_n$ stands for the Pochhammer symbol, indicating compound interest. The two major property functions cover a good number of mathematical occurrences being a much studied subject because of their rich structure and wide applications[6]

2. Importance of Summation Theorems

The summation theorems for instance, Kummer and Dixon's, are pivotal tools in hypergeometric series evaluation and also simplification. These identities provide formulae for summing a series that would be otherwise very difficult to evaluate and therefore occupy an extremely important position in both theoretical and applied mathematics. Hypergeometric series summing has great importance in solving differential equations, integrals and other problems of mathematical physics; for example, this is true in quantum mechanics and statistical mechanics [2].

2.1 Kummer's Summation Theorem.

Kummer's paper states that the hyperspherical function has a special solution.

${}_2F_1$ when the argument is 1 . The theorem states:

$${}_2F_1(a,b;1+a-b;1)=\frac{\Gamma(1+a-b)\Gamma(1+a)}{\Gamma(1+a-b+a)\Gamma(1)}=\frac{\Gamma(1+a-b)\Gamma(1+a)}{\Gamma(1+2a-b)},$$

where Γ denotes the Gamma function, a generalization of the factorial function. This theorem is particularly useful for simplifying expressions involving hypergeometric functions at specific points [5]

2.1.1 Proof and Derivation

The proof of Kummer's theorem is achieved through a series of operations on the series definition of the hypergeometric function alongside properties of the gamma function used. The proof can be revealed in different ways, ranging from the simplest examples to the more complicated analytical methods. Besides the fact that the combinatorial interpretation of the theorem provides clarity of its significance and applications [4], it is also insightful.

2.1.2 Applications of the Concept in the Hypergeometric Functions

Kummer's summation theorem represents one of the main methods made available to simplify hypergeometric expressions of a more complex nature. To illustrate, it offers the possibility of a decrease in the series occurring in the solution of any differential equation, and for an evaluation algorithm for integrals. Certain cases are the application of blanket in mathematical physics problems where it assists in finding out the derived forms of solutions [7].

2.2. Dixon's Summation Theorem

Dixon's summation theorem provides a summation formula for a specific type of well-poised hypergeometric series. The theorem is given by:

$${}_3F_2\left(\begin{matrix}a,1+a/2,b\\1+a/2-b,1+a\end{matrix};1\right)=\frac{\Gamma(1+a)\Gamma(1+a/2-b)}{\Gamma(1+a-b)\Gamma(1+a/2)}$$

where ${}_3F_2$ is a generalized hypergeometric function. Dixon's theorem is particularly useful for series where the parameters satisfy specific relations, allowing for significant simplifications [3].

Proof and Derivation

The proof of Dixon's theorem typically involves sophisticated techniques from the theory of special functions, including properties of the Gamma and Beta functions. Alternative proofs may employ complex analysis or integral representations of hypergeometric functions [2]

2.3 Applications in Hypergeometric Functions

Dixon's summation theorem finds applications in evaluating hypergeometric series that arise in various contexts, including combinatorial identities and integrals in mathematical physics. It is especially useful when dealing with series where traditional summation methods are cumbersome or inapplicable [4].

2.4 Double Series and Their Convergence

A double series is a series of the form:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}$$

where a_{mn} are the terms of the series indexed by two indices m and n. The convergence of a double series depends on the manner in which the terms are summed. Several types of convergence can be defined for double series, including absolute convergence, conditional convergence, and uniform convergence.

2.5 Definition and Convergence Criteria

1. ***Absolute Convergence***: A double series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}$ is said to be absolutely convergent if the series of absolute values $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a_{mn}|$ converges. Absolute convergence implies that the order of summation can be interchanged without affecting the sum:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a_{mn}|<\infty\Rightarrow\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}$$

2. ***Conditional Convergence***: An individual series of the double series is conditionally convergent if it converges but does not converge absolutely. The convergence of this series is more subtle and on which the summation is carried out leads to the sum of the series.

3. ***Uniform Convergence***: The double series conegurates uniformly when the function of partial sums forms a sequence of functions that converges uniformly. The meaning of this convergence is particularly important in understanding interchangeability of limits and summations as well.

The process of the convergence of double series can use some techniques which are obtained from convergence techniques for single series, like the comparison test, ratio test, and root test [8].

2.6 Application of Summation Theorems to Double Integrals

The theorems of Kummer and Dixon can be used to sum the double series by adjusting and simplifying them. These theorems mathematically represent particular series with some formulas and then generalised these formulas for more problems with two indices.

3. Transformations and Simplifications

Kummer's Theorem in Double Series: From Kummer's theorem about double series it can be taken that in one of series with hypergeometric function.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}}\frac{x^m y^n}{m!\,n!}$$

Using Kummer's theorem, we can sum the inner series if it matches the form required by the theorem, simplifying the overall expression:

$$\sum_{n=0}^{\infty}\frac{(b)_n}{(c)_{m+n}}\frac{y^n}{n!}= {}_2F_1(b, c-m; c; y)$$

Then, Kummer's theorem might provide a closed form for the inner sum, reducing the double series to a single series.

Dixon's Theorem in Double Series: Dixon's summation theorem is useful for double series where both indices appear in hypergeometric terms. For example:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(1+a/2)_n(b)_n}{(1+a/2-b)_n(1+a)_{m+n}}\frac{x^m y^n}{m!\,n!}$$

Applying Dixon's theorem can simplify the inner series when it matches the conditions specified by the theorem, thus simplifying the evaluation of the entire double series:

$$\sum_{n=0}^{\infty}\frac{(1+a/2)_n(b)_n}{(1+a/2-b)_n(1+a)_{m+n}}\frac{y^n}{n!}= {}_3F_2\left(\begin{matrix}1+a/2, b, a\\ 1+a/2-b, 1+a\end{matrix}; y\right)$$

If Dixon's theorem provides a closed form for this sum, the double series can be reduced accordingly [3]

4. Examples Involving Kummer and Dixon Theorems

1 *Example 1*: Application of Kummer's Theorem

Consider the double series:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(b)_n}{(c+m+n)_n}\frac{x^m y^n}{m!\,n!}$$

Using Kummer's theorem on the inner sum:

$$\sum_{n=0}^{\infty}\frac{(b)_n}{(c+m+n)_n}\frac{y^n}{n!}=(1-y)^{-b}$$

the double series simplifies to:

$$\sum_{m=0}^{\infty}\frac{(a)_m}{m!}x^m(1-y)^{-b}=(1-y)^{-b}\sum_{m=0}^{\infty}\frac{(a)_m}{m!}x^m=(1-y)^{-b}(1-x)^{-a}$$

Thus, the double series reduces to:

$$(1-y)^{-b}(1-x)^{-a}$$

2 *Example 2*: Application of Dixon's Theorem

Consider the double series:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(1+a/2)_n(b)_n}{(1+a/2-b)_n(1+a)_{m+n}}\frac{x^m y^n}{m!\,n!}$$

Using Dixon's theorem:

$$\sum_{n=0}^{\infty}\frac{(1+a/2)_n(b)_n}{(1+a/2-b)_n(1+a)_{m+n}}\frac{y^n}{n!}= {}_3F_2\left(\begin{matrix}1+a/2, b, a\\ 1+a/2-b, 1+a\end{matrix}; y\right)$$

the double series simplifies depending on the specific form of ${}_3F_2$ function.

These examples illustrate the power of summation theorems in reducing the complexity of double series, thereby making them more manageable and providing deeper insights into their properties.

5. Interrelationships Between Kummer and Dixon Theorems

Similarities and Differences

Kummer and Dixon summation theorems are both essential tools in the realm of hypergeometric functions, but they apply to different forms and conditions of series.

Similarities:

- **Both deal with hypergeometric series:** Kummer's and Dixon's theorems provide closed-form summations for specific types of hypergeometric series. Kummer's theorem addresses the ${}_2F_1$ series, while Dixon's theorem deals with ${}_3F_2$ series.

- **Use of Gamma functions:** Both theorems involve the Gamma function in their summations, which generalizes factorials and plays a key role in expressing the results.

- **Historical and mathematical importance:** Both theorems have historical significance and are frequently cited in mathematical literature for their utility in simplifying complex series [2]

Differences:

- **Scope of application:** Kummer's theorem is specific to ${}_2F_1$ hypergeometric functions when the argument is 1 . In contrast, Dixon's theorem applies to ${}_3F_2$ hypergeometric series under wellpoised conditions.
- **Conditions and constraints:** Kummer's theorem requires specific parameter relationships, while Dixon's theorem is applicable under the well-poised condition, where parameters satisfy certain balanced relations.
- **Types of problems addressed:** Kummer's theorem often simplifies problems involving ordinary differential equations and special function evaluations. Dixon's theorem, on the other hand, is particularly useful in evaluating series that arise in combinatorial problems and multivariable functions [3][5].
- solutions involve ${}_2F_1$ runctions. For instance, many second-order linear ODEs.

6. Situations Where One Theorem is More Applicable

Kummer's Theorem:

- **Ordinary Differential Equations (ODEs):** Kummer's theorem is particularly useful in solving ODEs where the solutions involve ${}_2F_1$ functions. For instance, many second-order linear ODEs can be solved using hypergeometric functions, and Kummer's theorem can simplify the resulting expressions [6].
- **Evaluation at Specific Points:** It is ideal for cases where the hypergeometric function needs to be evaluated at $z = 1$, reducing complex expressions to simpler forms.

Dixon's Theorem:

- **Combinatorial Identities:** Dixon's theorem is extremely powerful in combinatorics, where many identities involve sums that can be expressed in the form of ${}_3F_2$ series.
- **Multivariable Calculations:** It is more applicable in multivariable hypergeometric functions where the parameters are interrelated in a specific balanced manner, making the summation theorem useful for simplifying these higher-order terms [4]

7. Combined Applications

Cases Where Both Theorems are Used in Conjunction

In some advanced problems, both Kummer's and Dixon's theorems can be applied together to simplify hypergeometric series and double series.

1. Double Series Simplification:

Consider a double series that involves both ${}_2F_1$ and ${}_3F_2$ components. Using Kummer's theorem, one can simplify the ${}_2F_1$ part, and subsequently apply Dixon's theorem to the resulting ${}_3F_2$ part. For example, in a series of the form:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(b)_n}{(c+m+n)_m(d+m+n)_n}\frac{x^m y^n}{m!\,n!},$$

Kummer's theorem can simplify the ${}_2F_1$ sum with respect to m, and then Dixon's theorem can be used on the resulting ${}_3F_2$ series.

2. Transformations in Mathematical Physics.

In quantum mechanics and statistical mechanics, calculations often involve nested sums of hypergeometric series. Using both theorems can transform complex integrals and series into more manageable forms, facilitating easier computation and interpretation [7].

Impact on Simplifying Complex Hypergeometric Functions and Double Series

By using both Kummer’s and Dixon’s theorems together, one can considerably simplify the job of evaluating hypergeometric functions and double series. This dual approach

Improves Computational Efficiency: Instead of perplexing series to become simpler, closed-form expressions are derived. With this approach, we can treat big problems without wasting computing resources and time [1].

Enhances Analytical Insight: By simplifying these series, one is able to delve deeper into the issue concerning the properties as well as the behavioral aspect of the functions involved. It simplifies the work with the limit processes and such values as zeros.

Facilitates Further Research: The new intuitions and methods developed with the help of these theorems pave the way for more comprehensive research, including mathematical physics, combinatorics, and advanced calculus, whose research is greatly facilitated by hypergeometric series [4].

8. Advanced Applications

Extension of Multivariable Hypergeometric Series

Generalization to Multiple Variables

While the generalized hypergeometric functions of multiple variables, for example the Lauricella, Appell, and Kampé de Fériet functions, expand the concept of classical hypergeometric functions to more than one variable. These schools of multivariable hypergeometric functions emerge naturally in different areas such as mathematical physics, engineering, and number theory.

1 **Lauricella Functions:** These are generalizations of the Gauss hypergeometric function to multiple variables. The Lauricella F_D function is defined as:

$$F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n} m_1!\cdots m_n!} x_1^{m_1}\cdots x_n^{m_n}$$

The Lauricella function appears whenever string theory is discussed, in a specific context of string amplitudes computation [11].

2 **Appell Functions:** These functions are two-variable generalizations of the hypergeometric function ${}_2F_1$. There are four Appell functions, denoted as F_1, F_2, F_3, and F_4. For instance, F_1 is defined as:

$$F_1(a; b_1, b_2; c; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n} m!\, n!} x^m y^n$$

Appell functions are used for electromagnetics problem solving, namely potential theory and wave propagation in anisotropic media [4].

3 **Kampé de Fériet Functions:** These are further generalizations involving more parameters and variables. They take the form:

$$F_{(r:s)}^{(p:q)}\begin{bmatrix} a_1, \ldots, a_p: & b_1, \ldots, b_q; & c_1, \ldots, c_r \\ d_1, \ldots, d_s; & e_1, \ldots, e_t; & f_1, \ldots, f_u \end{bmatrix}; x_1, \ldots, x_n$$

Kampé de Fériet Functions in Fluid Dynamics: These apply in the analysis of complicated fluid motion and in the solution of partial differential equations emerging in the theory of fluid dynamics [16].

9. Role in Mathematical Physics

Applications in Quantum Mechanics and Statistical Mechanics

Quantum Mechanics: Hypergeometric functions frequently appear in the solutions of the Schrödinger equation for various potential models. For example, the radial part of the wavefunction for the hydrogen atom is expressed in terms of the confluent hypergeometric function ${}_1F_1$ [10]

Statistical Mechanics: With hypergeometric functions in statistical mechanics, partition functions and correlation functions are defined by them. To illustrate, the one-dimensional Ising model partition function can be described with hypergeometric series [9].

***Field Theory*:** In the field of quantum field theory hypergeometric functions are used to determine Particle interactions Feynman diagrams and to measure amplitudes. Especially Dixon's summation theorem assists in easy representation of series that appear during such operations [15].

10. Specific Examples

Hydrogen Atom: The radial Schrödinger equation for the hydrogen atom reduces to a confluent hypergeometric differential equation, with solutions involving ${}_1F_1$:

$$R_{nl}(r) = e^{-r/n} r_1^l F_1(l - n + 1; 2l + 2; 2r/n)$$

Ising Model Partition Function: The partition function Z for the 1D Ising model is given by:

$$Z = \sum_{k=0}^{N} \binom{N}{k} \exp\left(-\frac{2kJ}{k_B T}\right)$$

where $\binom{N}{k}$ can be expressed in terms of hypergeometric series [9].

11. Algorithmic Approaches to Applying Theorems

While being confronted with the applications of summation theorems like the ones of Kummer's and Dixon's in computational settings, one must use algorithms with high speed and efficiency to handle series and integrals of large size.

1. *Symbolic Computation*: Mathematica, Maple and other such software have the capacity to use the hypergeometric series, but their application of the Kummer's theorem and Dixon's theorem remains symbolic.

2. *Numerical Methods*: However, numerical techniques are applied when the symbolic method is found to be impractical. They come in and substitute the hypergeometric series by an approximation computation. As for the methods, you may encounter those that necessitate series truncation, adaptive integration, and convergence acceleration.

Software and Tools

1. *Maple*: Maple covers an advanced spectrum of special functions, including hypergeometric seriesIt consists of the following tools: series transformations and expansions using summation theorems and their generalization [12].

2. *Matlab*: As for Matlab, its Symbolic Math Toolbox contains functions for summation of hypergeometric series which is done with the built-in functions. This method shows its exceptional strength for engineering problems that are solved in a numerical form [13].

3. *Python (SymPy):* SymPy, a Python library which works with symbolic mathematics, uses hypergeometric functions and it is possible to make summation theorems programmatically. It is free from copyrights and connects seamlessly with other scientific programming libraries [15].

RESULT AND DISCUSSION

In this exploration, we delved into the applications of Kummer and Dixon summation theorems within the realm of hypergeometric functions. Kummer's theorem furnishes a closed-form expression for the sum of a ${}_2F_1$ hypergeometric series under specific conditions, while Dixon's theorem extends this capability to ${}_3F_2$ hypergeometric series, providing analogous closed-form summations in select scenarios. This theorem provides a tool for the transformation and simplification of double series; thus, it allows us to analyze the convergence of a series. Furthermore, we demonstrated a comparative study explaining the similarity and dissimilarity of the concepts described by Kummer and Dixon theorems and illuminating when one theorem may be more suitable for the other. Also, we studied how we could utilize them in conjunction that emphasized how they function as a team to this effect. Moreover, a retrospection of these trends was undertaken which comprises generalizations of these theorems, their significance in mathematical physics and computational methodology since they provided the knowledge of those circumstances. This research with its multivariable extensions, applications in quantum mechanics providing new answers as well as statistical mechanics, and the development of new computational techniques involving processes that are still happening present a promising future in this exciting branch.

FUTURE WORK

Extensions of original Kummer and Dixon theorems on multivariable hypergeometric functions may help us broaden the frontiers of mathematics and find new avenues of partnerships between science and other disciplines. Through the generalisation of these theorems to multidimensional cases, we go deeper into theory and improve the usability of methodology in the range of new scientific endeavors. Not only this could be an essential extension of hypergeometric functions, but also this could find practical applications in the areas of quantum mechanics, statistical mechanics and maybe in the others. It is of paramount importance to bring about further inventions in computation techniques and software as well as symbolic and numerical machinery which deals with hypergeometric series. The process requires the improvement of algorithms, software intelligence extension, and integrating highly engaged functions to process tedious tasks. It is expected that the improved computational tools would make it easier to be applicable to more problems by applying these theorems, enabling the enlargement of the landscape of the possible problems for more efficient and accurate solution. Among others, fostering collaborations between mathematicians, physicists, computer scientists, and engineers can be one of the best ways to get interdisciplinary fuels and new perspectives. So, the combination of research experts of interdisciplinary origin could facilitate breakthrough solutions to mysteries and enhance speed of engineering. Interdisciplinary collaboration brings experts from varied fields on a common platform to contribute their distinctive views and knowledge to the domain of hypergeometric functions. Such convergence of minds and methods goes a long way in propelling the boundaries of science and technology.

Kummer and Dixon summation theorems constitute a key element in the theory of hypergeometric functions. Having established the starting point as well as a strong base for future application of these functions, they open unexplored possibilities and deepen the understanding to a whole new dimension.

CONCLUSION

In the pursuit of this exploration, we unearthed the acknowledgement of Kummer and Dixon summation theorem in the study of hypergeometsic functions. The Kummer's theorem provides a closed form expression for the hypergeometric series of type ${}_2F_1$ with the following condition, while Dixon's theorem extends the validity of the theorem to the special case of ${}_3F_2$ hypergeometric series. The theorem then furnishes analogous closed form summations. Needless to say, these theorems constitute the pillars behind which the transformation into and simplification of the double series stands, which in its turn permits the scholars to analyse and evaluate the series more efficiently. Further, we carried out a comparative analysis which revealed much about Kummer and Dixon theorems' similarities and differences, features that show when one theorem perhaps, is more applicable than the other. Another important topic we had at hand was the ramifications of these theorems by themselves, as well as the combined ways in which they could be used additively in a simplistic way to simplify differential equations and series containing hypergeometric functions. On the other hand, generalizations of these theorems, their significance in math at a more advanced levels of physics, and the progression on computational methods were evaluated. Such a progress takes many forms, among them, several extensions of multi-variable theories, implementation of these concepts to quantum mechanics and statistics, and development of advanced computational techniques, and in so doing this field will attract many more students and researchers alike.

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