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Mittag-Liffler Functions and Applications

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Abstract - Mittag-Leffler functions, introduced by Gösta Mittag-Leffler in 1903, play a pivotal role in fractional calculus and numerous applied sciences. They generalize exponential functions and are characterized by their rich structure, which enables modeling of processes exhibiting memory and hereditary properties. The Mittag-Leffler function emerges as a natural solution to fractional differential equations, making it invaluable in areas such as viscoelasticity, anomalous diffusion, and control theory. Recent advancements have extended its applications to stochastic processes, bioengineering, and mathematical physics. This paper explores the fundamental properties, analytical behavior, and diverse applications of Mittag-Leffler functions, highlighting their importance in solving complex real-world problems.

Keywords: Mittag-Leffler functions, fractional calculus, fractional differential equations, anomalous diffusion

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INTRODUCTION

The Mittag-Leffler function, named after the Swedish mathematician Gösta Mittag-Leffler, is a generalization of the exponential function and a cornerstone in fractional calculus and complex analysis. First introduced in 1903, it was originally developed to study series expansions and later found extensive applications in modeling non-exponential relaxation phenomena in various scientific disciplines. The function is defined mathematically as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},$$

where z is a complex number, α>0 controls the growth rate, and β shifts the argument of the gamma function. Special cases of the Mittag-Leffler function include $E_{1.1}(z) = e^z$, which reduces to the classical exponential function, and $E_{2.1}(z)$, which is integral to solving fractional telegraph equations. This versatility highlights its relevance in both classical and fractional systems.

One of the most remarkable properties of the Mittag-Leffler function is its entire nature, being holomorphic across the entire complex plane. It converges for all complex z when α>0, and its behavior for large |z| can be described using asymptotic expansions. For instance, when α=1, the function aligns with the exponential decay e^{-z}. It also admits an integral representation, given by

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_C \frac{t^{\alpha-\beta} e^t}{t^\alpha - z} dt,$$

is a contour in the complex plane. This integral formulation provides insights into its deep connections with other special functions and allows for analytical continuation in various contexts.

The Mittag-Leffler function's utility extends far beyond pure mathematics. In fractional calculus, it is a fundamental solution to fractional-order differential equations, enabling the modeling of systems with memory effects, such as viscoelastic materials. In physics, it describes anomalous diffusion processes, including sub-diffusion and super-diffusion, which deviate from classical Brownian motion. The function is also pivotal in control theory, where it models non-linear dynamics in fractional-order systems, and in signal processing, aiding in the analysis of systems with power-law characteristics. Furthermore, its role in relaxation and oscillation phenomena is significant in materials science and dielectric studies, where it describes non-exponential relaxation and oscillatory behavior.

Numerical computation of the Mittag-Leffler function requires careful consideration due to its complex nature. Series expansions are commonly used for small arguments, while asymptotic approximations are employed for large arguments. Additionally, numerical techniques involving contour integrals are applied to evaluate the function in specific scenarios. Modern computational tools, including MATLAB and Python libraries, provide efficient implementations for evaluating the Mittag-Leffler function across various parameter ranges. The function has also been generalized into multi-parameter and matrix forms, allowing for greater flexibility in addressing complex systems and equations, such as matrix Mittag-Leffler functions for matrix differential equations.

The Mittag-Leffler function's importance lies in its ability to bridge classical mathematical concepts with emerging fields, making it indispensable in theoretical and applied sciences. From its historical origins in complex analysis to its modern applications in fractional calculus, physics, and engineering, the Mittag-Leffler function continues to be a vital tool in understanding and modeling complex systems. Its ability to generalize the exponential function, coupled with its rich analytical properties and broad applicability, underscores its significance in contemporary mathematics and science.

PRELIMARIES

Riemann-Liouville Fractional Derivative The classical Riemann-Liouville fractional derivative [132] of order μ *is* usually defined by

$$D^{\mu}\left[f(x)\right]=\frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}dt,\quad Re(\mu)<0. \quad \text{.....1}$$

where the integration path is a line from 0 to x in the complex t-plane. For the case (m-1)< Re (μ)<m(m=1,2,3,...),.. it is defined by

$$D^{\mu}\left[f(x)\right]=\frac{d^m}{dx^m}D^{\mu-m}\left\{f(x)\right\}$$
$$=\frac{d^m}{dx^m}\left[\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}dt\right], \quad \text{.....2}$$

The extended Riemann-Liouville fractional derivative operator was defined by Özarslan and Özergin as follows:

$$D^{\mu,p}\left[f(x)\right]=\frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}\exp\left(\frac{-px^2}{t(x-t)}\right)dt,$$
.....3

In the above

$\mathrm{Re}(\mu)<0,\mathrm{Re}(p)>0\ \mathrm{for}\ (m-1)<\mathrm{Re}(\mu)<m\ (m=1,2,3,...)$
and

$$D^{\mu,p}\left[f(x)\right]=\frac{d^m}{dx^m}D^{\mu-m,p}\left\{f(x)\right\}$$
$$=\frac{d^m}{dx^m}\left[\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}\exp\left(\frac{-px^2}{t(x-t)}\right)dt\right], \quad \text{....4}$$

where the path of integration is a line from 0 to x in the complex t-plane. For the case p→0 ,we obtain the classical Reimann-Liouville fractional derivative operator.

Again Özarslan and Yilmaz extended the Mittag-Leffler function defined by

Let $\alpha,\beta,\gamma,c,p\in C,\mathrm{Re}(p)>0;\mathrm{Re}(c)>0,\mathrm{Re}(\gamma)>0$, then

$$E_{\alpha,\beta}^{\gamma;c}(x;p)=\sum_{k=0}^{\infty}\frac{B_p(\gamma+k,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_k}{\Gamma(\alpha k+\beta)}\cdot\frac{x^k}{k!}, \quad \text{.....5}$$

where $B_p(x,y)=B(x,y;p)$

$$=\int_0^1 u^{x-1}(1-u)^{y-1}\exp\left[\frac{-p}{u(1-u)}\right]du,$$
$$\mathrm{Re}(p)>0;\mathrm{Re}(x)>0,\mathrm{Re}(y)>0. \quad \text{......6}$$

is the prolonged Euler's Beta function demarcated in [25]

Above Mittag-Leffler function can be derived by using the following relation given by Chaudhary et al., Chaudhary and Zubair and

$$\frac{(\gamma)_k}{(c)_k}=\frac{B(\gamma+k,c-\gamma)}{B(\gamma,c-\gamma)} \quad \text{.......7}$$

Part-1:

In this section, we introduce and study the following extended form of Mittag-Lefflerfunctiondefined as follows:

$$E_{\alpha,\beta}^{(\gamma,c);q}(x;p)=\sum_{n=0}^{\infty}\frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_{nq}}{\Gamma(\alpha n+\beta)}\cdot\frac{x^n}{n!}, \quad \text{.......8}$$

$\alpha,\beta,\gamma,p\in C,\mathrm{Re}(p)>0;\mathrm{Re}(\alpha)>0,\mathrm{Re}(\gamma)>0,\mathrm{Re}(\beta)>0,$

$\mathrm{Re}(c)>0,q<\mathrm{Re}(\alpha)+1.$

and $\dfrac{(\gamma)_{nq}}{(c)_{nq}}=\dfrac{B(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)}$, ...9

Foremost Outcomes:

Theorem: 1

Let $\mathrm{Re}(p)>0,\ \mathrm{Re}(\mu)>0,\ \mathrm{Re}(\lambda)>0,$

$\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,$ *where* $q<\mathrm{Re}(\alpha)+1,$ *then*

$$D^{\lambda-\mu,p}\left[x^{\lambda-1}E_{\alpha,\beta}^{\mu,q}(x^q;p)\right]=\frac{x^{\mu-1}}{\Gamma(\mu-\lambda)}B(\lambda,\mu-\lambda)\,E_{\alpha,\beta}^{(\lambda,\mu)}(x^q;p) \quad \text{......10}$$

Proof:

Using the definition of extended Riemann-Liouville fractional derivative given by (3) in the left hand side of (10) and substituting the value of Mittag Leffler type function from the equation mentioned in the introductory section, we find that

$$D^{\lambda-\mu,p}\left[x^{\lambda-1}E_{\alpha,\beta}^{\mu,q}\left(x^q;p\right)\right]$$
$$=\frac{1}{\Gamma(\mu-\lambda)}\int_0^x t^{\lambda-1}E_{\alpha,\beta}^{\mu,q}\left(t^q\right)(x-t)^{-\lambda+\mu-1}\exp\left[\frac{-p\,x^2}{t(x-t)}\right]dt,$$
$$=\frac{1}{\Gamma(\mu-\lambda)}\int_0^x t^{\lambda-1}\sum_{k=0}^{\infty}\frac{(\mu)_{kq}}{\Gamma(\alpha k+\beta)}\frac{(t)^{kq}}{k!}(x-t)^{-\lambda+\mu-1}\exp\left[\frac{-p\,x^2}{t(x-t)}\right]dt,$$

Moving the order of summation and integration and stroking t = xu , we obtain

$$=\frac{x^{\mu-1}}{\Gamma(\mu-\lambda)}\sum_{k=0}^{\infty}\frac{(\mu)_{kq}}{\Gamma(\alpha k+\beta)}\frac{x^{kq}}{k!}\int_0^1 u^{\lambda+kq-1}(1-u)^{-\lambda+\mu-1}\exp\left[\frac{-p}{u(1-u)}\right]du,$$
$$=\frac{x^{\mu-1}}{\Gamma(\mu-\lambda)}\sum_{k=0}^{\infty}\frac{(\mu)_{kq}}{\Gamma(\alpha k+\beta)}\frac{x^{kq}}{k!}B_p(\lambda+kq,\mu-\lambda)$$

By substituting the equation 8 in the above one we can get the desired results.

Differentiation formula:

Theorem-2:

For the extended MittagLeffler function, the following differentiation formula holds:

$Let\,\mathrm{Re}(c)>0, q<\mathrm{Re}(\alpha)+1,$

$$\frac{d^n}{dx^n}\left[x^{\beta-1}E_{\alpha,\beta}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right)\right]=x^{\beta-n-1}E_{\alpha,\beta-n}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right) \quad \text{……11}$$

Proof:

We prove this theorem by applying principle of mathematical induction method. We start with n=1

$$\frac{d}{dx}\left[x^{\beta-1}E_{\alpha,\beta}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right)\right]$$
$$=\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_{nq}}{\Gamma(\alpha n+\beta)}\cdot\frac{(\lambda)^n(x)^{\alpha n+\beta-1}}{n!}\right),$$
$$=\left(\sum_{n=0}^{\infty}\frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_{nq}}{\Gamma(\alpha n+\beta)}\frac{(\lambda)^n(\alpha n+\beta-1)(x)^{\alpha n+\beta-2}}{n!}\right) \quad \text{……12}$$

$$\Rightarrow\frac{d}{dx}\left[x^{\beta-1}E_{\alpha,\beta}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right)\right]$$

$$=\left(\sum_{n=0}^{\infty}\frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_{nq}}{\Gamma(\alpha n+\beta-1)}\frac{(\lambda)^n(x)^{\alpha n+\beta-2}}{n!}\right)$$

Therefore,

$$\frac{d}{dx}\left[x^{\beta-1}E_{\alpha,\beta}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right)\right]=x^{\beta-2}E_{\alpha,\beta-1}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right) \quad \text{…..13}$$

Therefore, the result from equation 11 is true for n=1

Then let assume the result in 12 equation is true for n=k , which is

$$\frac{d^{k+1}}{dx^{k+1}}\left[x^{\beta-1}E_{\alpha,\beta}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right)\right]$$
$$=\frac{d}{dx}\left(\frac{d^k}{dx^k}\left[x^{\beta-1}E_{\alpha,\beta}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right)\right]\right)$$
$$=\frac{d}{dx}\left(x^{\beta-(k+1)}E_{\alpha,\beta-k}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right)\right),$$

By using equation 13, we can have

$$\frac{d^{k+1}}{dx^{k+1}}\left[x^{\beta-1}E_{\alpha,\beta}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right)\right]$$
$$=\left(x^{\beta-(k+2)}E_{\alpha,\beta-(k+1)}^{(\gamma,c);q}\left(\lambda\,x^{\alpha};p\right)\right)$$

As the result is true for the value of n= k+1, henceforth the theorem is true for all n ∊ N

Theorem-3:

$Let\ \alpha,\beta,\gamma,c,p\in C,\mathrm{Re}(p)>0;\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0,$

$\mathrm{Re}(c)>0, q<\mathrm{Re}(\alpha)+1,$

$$\frac{d^n}{dx^n}\left[E_{\alpha,\beta}^{(\gamma,c);q}(x;p)\right]=(\gamma)_{qn}E_{\alpha,\beta+n\alpha}^{(\gamma+nq,c+nq);q}(x;p) \quad \text{…...14}$$

Proof:

Lets start with n= 1 in equation 14, we obtain

$$\frac{d}{dx}\left[E_{\alpha,\beta}^{(\gamma,c);q}(x;p)\right]=\left(\sum_{n=0}^{\infty}\frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_{nq}}{\Gamma(\alpha n+\beta)}\cdot\frac{nx^{n-1}}{n!}\right) \quad \text{…..15}$$

Substituting, n=m+1 in equation 15 and using the pochhammer symbol property and gamma function, we have

$$\frac{d}{dx}\left[E_{\alpha,\beta}^{(\gamma,c);q}(x;p)\right]=\sum_{m=0}^{\infty}\frac{B_p(\gamma+(m+1)q,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_{(m+1)q}}{\Gamma(\alpha(m+1)+\beta)}\cdot\frac{x^m}{m!}$$

$$\frac{d}{dx}\left[E_{\alpha,\beta}^{(\gamma,c);q}(x;p)\right]=(\gamma)_q E_{\alpha,\beta+\alpha}^{(\gamma+q,c+q);q}(x;p) \quad \text{…..16}$$

By differentiating the equation 16 we will have the following

$$\frac{d}{dx}\left[E_{\alpha,\beta}^{(\gamma,c);q}(x;p)\right]=\frac{d}{dx}\left[(\gamma)_q E_{\alpha,\beta+\alpha}^{(\gamma+q,c+q);q}(x;p)\right]$$
$$=(\gamma)_q\left(\sum_{n=0}^{\infty}\frac{B_p(\gamma+q+nq,c-\gamma)}{B(\gamma+q,c-\gamma)}\frac{(c+q)_{nq}}{\Gamma(\alpha n+\beta+\alpha)}\cdot\frac{nx^{n-1}}{n!}\right) \quad \text{…..17}$$

Substituting n = s+1 and using the pochhammer symbol property and gamma function in equation 17

We get

$$= (\gamma)_q \left(\sum_{s=0}^{\infty} \frac{B_p(\gamma+2q+sq, c-\gamma)}{B(\gamma+q, c-\gamma)} \frac{(c+q)_{(s+1)q}}{\Gamma(\alpha(s+1)+\beta+\alpha)} \cdot \frac{nx^s}{n!} \right)$$

$$= (\gamma)_q \left(\sum_{s=0}^{\infty} \frac{B_p(\gamma+2q+sq, c-\gamma)}{B(\gamma+q, c-\gamma)} \frac{(c+q)_{(s+1)q}}{\Gamma(\alpha(s+1)+\beta+\alpha)} \cdot \frac{nx^s}{n!} \right) \frac{B(\gamma+2q, c-\gamma)}{B(\gamma+2q, c-\gamma)}$$

Therefore, we get

$$\frac{d^2}{dx^2}\left[E_{\alpha,\beta}^{(\gamma,c);q}(x;p) \right] = (\gamma)_{2q} \, E_{\alpha,\beta+2\alpha}^{(\gamma+2q,c+2q);q}(x;p),$$

By iteratively applying this process n times, we arrive at the desired outcome.

Analysis of Mittag-Leffler Functions and Applications:

The Mittag-Leffler function has emerged as an essential component in the fields of mathematical analysis and applied sciences. It offers a complete framework that can be used to address complex problems that exceed the limitations of traditional models. This function was first proposed by Gotta Mittag-Leffler in the early 20th century. It is articulated in a manner that extends the exponential function and creates a major interaction with fractional calculus for the first time. Specifically, the mathematical formulation emphasises on characteristics that allow for interpolation between exponential and power-law behaviours. As a result, it is especially effective at characterising phenomena that exhibit memory effects, anomalous diffusion, and non-linear dynamics. A great amount of attention has been paid to the Mittag-Leffler function in a variety of fields, such as physics, engineering, control theory, and computational mathematics. This is due to the extraordinary adaptability of the function as well as the richness of its analytical properties.

One of the most notable characteristics of the Mittag-Leffler function is its ability to fulfil the role as a connector between fractional and classical systems. When it comes to correctly describing real-world phenomena that include memory or hereditary traits, conventional frameworks that are based on integer-order calculus usually presume exponential growth or decay processes. This may render these frameworks inadequate. On the other hand, the Mittag-Leffler function provides a solid foundation for solving fractional differential equations since it is inherently intended to deal with the complications that are involved. Because of this, it is an essential tool in fractional calculus, which is a branch of mathematics that extends the scope of classical calculus by expanding the concepts of derivatives and integrals to include orders that are not integers. In these kinds of situations, the Mittag-Leffler function develops as a natural extension of the exponential function. It is seen in the solutions of differential equations that reflect a wide variety of engineering and physical physical events.

For example, the Mittag-Leffler function is able to handle a wide variety of parameter options, which have a significant impact on the features and applications of the function. This demonstrates the function's adaptability. Using these parameters, the function is able to represent a wide range of processes, from quick decay to protracted tail characteristics. It is capable of simulating all of these processes. Because of its amazing adaptability, it has been able to be used in the investigation of anomalous diffusion, which is a phenomena that occurs in complex systems such as biological tissues, porous media, and financial markets. Anomalous diffusion is distinct from the conventional Gaussian diffusion, exhibiting features of either sub-diffusion or super-diffusion. These traits are effectively described by fractional-order equations that include the Mittag-Leffler function. Its relevance in the modelling of complex system dynamics is highlighted by the fact that it is able to articulate processes that depart from standard diffusion rules.

The Mittag-Leffler function demonstrates its analytical skills in a number of different domains, one of which is the description of relaxation and oscillatory events. This particular function is used to describe viscoelastic behaviour in the field of materials science. This is a kind of behaviour in which materials exhibit properties that are characterised by both elasticity and viscosity. The Mittag-Leffler function, in contrast to conventional models, which make use of exponential functions to characterise stress-strain relationships, successfully includes the power-law characteristics that are inherent in viscoelastic reactions. Because of this, it is a very useful tool for understanding materials that exhibit behaviours that are reliant on memory. In a similar fashion, this function performs the function of characterising non-exponential relaxation within the realm of dielectrics. This is a phenomenon that is frequently observed when the polarisation of a dielectric substance that is subjected to an electric field decreases at a rate that is more gradual than what conventional exponential models would suggest. These kinds of applications highlight the relevance of this concept in terms of clarifying the complex dynamics of systems that are marked by fundamental memory effects.

The use of the Mittag-Leffler function extends beyond the confines of the physical sciences, and it also finds importance in the fields of engineering and control theory. In order to model system dynamics that are marked by memory or hereditary qualities, fractional-order systems commonly make use of the Mittag-Leffler function. These systems may be thought of as an extension of the ideas that are used in conventional integer-order systems. When it comes to the field of control theory, fractional-order controllers have shown to be more effective than their standard counterparts when it comes to dealing with complex systems. The controllers in question make use of the Mittag-Leffler function in order to develop control techniques that improve both stability and performance. These controllers expand the capabilities of proportional-integral-derivative controllers. As a result of the function's ability to express fractional dynamics, it has become an important component in the fields of robotics, signal processing, and process control. More specifically, it has considerably enhanced the responsiveness of the systems to a wide variety of inputs and disturbances.

Beyond the many applications it may be used for, the Mittag-Leffler function has outstanding mathematical qualities that contribute to its versatility. The fact that this function is complete, which indicates that it is holomorphic throughout the whole complex plane, ensures that it can be managed analytically. At the same time as the asymptotic qualities provide useful insights into the behaviour of systems at large scales, the convergence for all complex arguments guarantees the stability of mathematical calculations. In addition to this, the function may be expressed in integral form, which not only improves its analytical continuation but also establishes linkages to other special functions. Due to the properties of this topic, it is unusually well-suited for addressing problems that are associated with complex systems, particularly in the field of fractional calculus.

It is important to take into account the computational dimensions of the Mittag-Leffler function since its use in real-world circumstances typically involves numerical difficulties. In spite of the fact that the series representation of the function converges well for smaller arguments, evaluating it for bigger arguments requires the use of various approaches. In order to address these issues, numerical approaches such as contour integration and asymptotic approximations have been developed. These techniques ensure accurate calculation over a wide variety of contexts. Advanced methods for the Mittag-Leffler function have been included into modern computational instruments, such as software libraries in MATLAB, Python, and R. This has resulted in the function's increased use across a wide range of fields. Because of recent advancements in numerical computing, the function is now more accessible to scholars as well as practitioners, which has resulted in a major expansion of its significance.

As a result of the Mittag-Leffler function's role as a catalyst for various generalisations, it has been able to meet the needs of systems that are becoming more complex. Furthermore, multi-parameter versions provide additional degrees of freedom, which makes it possible for the function to reflect a more comprehensive range of behaviours. The expansion to matrix arguments, in a similar fashion, makes its use in linear systems of differential equations easier to accomplish. In these systems, the solutions are articulated using matrix Mittag-Leffler functions. The generalisations that have been discussed above highlight the extraordinary versatility of the function as well as its ability to face more complex mathematical and physical issues.

The Mittag-Leffler function is a perfect example of the deep interaction that exists between theoretical and practical mathematics. This function is located within the wide world of mathematical analysis. This was first motivated by theoretical studies in complex analysis; however, its usefulness has since expanded to include practical issues in the fields of physics, engineering, and computer science. The original drive for its creation was derived from these discoveries. The existence of this duality underscores the relevance of developing mathematical tools that improve theoretical knowledge while also providing answers to problems that are encountered in practice. This philosophical approach is shown by the Mittag-Leffler function, which serves as a bridge between mathematical theory and practical application. It is distinguished by its complex analytical framework and vast applications.

The inquiry that is now being conducted into the Mittag-Leffler function and its expansions gives a tremendous opportunity for future advancements in the fields of science and technology. As more research is conducted into fractional calculus and the many applications of this mathematical concept, the function's relevance in modelling complex systems is likely to increase. Because of its ability to contain the dynamics of systems that are defined by memory effects, non-linear responses, and anomalous behaviours, it has established itself as an important tool for studying the intricacies of both the natural and artificial environments. Additionally, advancements in computing techniques and software innovation will lead to an increase in its availability, which will make it easier to incorporate it into new technologies and scientific achievements.

In conclusion, the Mittag-Leffler function is a prime example of the astonishing potential of mathematical creativity to be able to handle complex problems. In modern mathematics and applied sciences, it has become an indispensable tool due to the fact that it is both flexible and precise in its analytical capabilities. The investigation of fractional calculus, which extends into control theory and other areas, demonstrates that the function is able to go beyond conventional ideas and accommodate a wide range of applications, so confirming that it will continue to be significant in the future. As the fields of science and technology continue to grow, the Mittag-Leffler function will undoubtedly continue to be an essential component in mathematical investigation and application. This will push breakthroughs in our understanding of the complexities of our universe and in our ability to express them.

Application Area:

An intriguing mathematical construction, the Mittag-Leffler function has evolved into a flexible tool for tackling issues in many different scientific fields. Essential to modelling, simulation, and analysis, it generalises the exponential function and has strong linkages to fractional calculus. The Mittag-Leffler function has shown to be very versatile and useful throughout its history, from its inception in pure mathematics to its many uses in the practical sciences. Mathematical theory and computational methods have also advanced in response to this function, making them more equipped to handle complex systems and ever-changing surroundings.

The Mittag-Leffler function is useful because it may characterise processes that behave in an abnormal or non-exponential way. Classical functions, such as the exponential function, often operate under idealised circumstances that may not adequately represent the intricacies found in actual systems. The Mittag-Leffler function fills this need by allowing tuning parameters to characterise a broad variety of behaviours. Memory effects, fractional dynamics, and power-law distributions are the most common types of phenomena where its adaptability is on display. Materials science, biology,

conomics, and engineering are just a few of the many fields that often deal with them.
When it comes to modelling anomalous diffusion, a process that doesn't follow the expected Gaussian pattern from standard diffusion equations, the Mittag-Leffler function has become an indispensable tool in the physical sciences. Financial markets, turbulent flows, biological tissues, and many other complex systems display non-Gaussian behaviours, such as lengthy correlations or heavy tails. The fast initial changes and the long-term tails of diffusion processes may be described by the Mittag-Leffler function, which provides a strong foundation for these systems. This function is great at capturing the subtleties of sub- and super-diffusion, which defy standard diffusion models, since its parameters are quite adaptable.

The function's capacity to simulate memory-based systems is shown by its involvement in viscoelasticity. The reactions of viscoelastic materials, like polymers, combine elasticity with viscosity. Since the stress-strain behaviour of these materials is dependent on their deformation history, exponential functions cannot adequately characterise them. A better model for similar behaviours is the Mittag-Leffler function, which has parameters that can be adjusted and a power-law decay. Similarly, in dielectric relaxation, the function describes the microscopic principles behind the slower-than-exponential loss of polarisation in materials subjected to electric fields.

The Mittag-Leffler function is not only important in fractional-order systems in control theory, but it is also employed extensively in the physical sciences. The integer-order dynamics upon which traditional control systems are built presupposes linearity and instantaneity of system reactions. On the other hand, memory effects and delayed reactions are common features of many non-linear systems seen in the actual world. To better manage such complexity, fractional-order controllers are used, which include the Mittag-Leffler function. The characteristics of the function allow for accurate modelling and control of dynamic systems, making these controllers particularly useful in robotics, process automation, and signal processing.

Its generalisability is the mathematical beauty of the Mittag-Leffler function. This function guarantees stability and convergence in analytical and numerical applications since it is holomorphic throughout the whole complex plane as an entire function. Integral representations allow for deeper theoretical inquiry via links to other mathematical constructions, while asymptotic behaviour offers insights into large-scale system dynamics. Because of its many useful features, it is a powerful instrument for solving problems in computer science and applied mathematics.

The domain of fractional differential equations is one where the Mittag-Leffler function is most often used. These equations allow for fractional dynamical system modelling by expanding classical differential equations to include derivatives of non-integer orders. As a cornerstone of fractional calculus, the Mittag-Leffler function often shows up as a solution to these kinds of problems. This function highlights its significance in characterising systems displaying non-linear damping, long-range dependencies, memory effects, and other abnormal behaviours.

There has been continuous study on numerically computing the Mittag-Leffler function because of its relevance in real-world applications. Its series representation is simple for simple inputs, but evaluating the function for complex values or large arguments might be difficult because of numerical instability or sluggish convergence. Asymptotic approximations, numerical integration approaches, and specialised algorithms have been created by academics to address these challenges. These developments have allowed fast and accurate computation of the Mittag-Leffler function even for complicated cases. Its accessibility to academics and practitioners in several domains has been enhanced by the availability of implementations in computational software packages, which further facilitates its usage.

The original version of the Mittag-Leffler function is not the only one that has an influence. Its applicability has been broadened to include more complex systems due to its generalisations. Mittag-Leffler functions with more than one parameter may simulate more complex processes since their behaviour can be fine-tuned. Matrix Mittag-Leffler functions, which provide solutions expressed in terms of matrix arguments, were also created for use in systems of linear equations. These extensions show how flexible the function is and how it may be used to solve new problems in mathematical modelling and practical research.

Modelling enzyme reactions, gene expression, and neural network dynamics are some of the computational biological processes that have made use of the Mittag-Leffler function. The complex behaviours of biological systems, which are often out of equilibrium, are well captured by its capacity to characterise fractional dynamics and memory effects. Market dynamics, risk analysis, and option pricing are all models that use the function in economics and finance. Systems with heavy-tailed distributions or long-term correlations may be better understood using its flexibility and power-law properties.

One cannot emphasise enough how important the Mittag-Leffler function is in engineering. Since the introduction of fractional-order models, their applications in communications, system identification, and signal processing have grown substantially. Both the precision and the adaptability of descriptions of system behaviour are enhanced by these models, which expand upon conventional methods. Developing sophisticated engineering systems and algorithms has been greatly facilitated by the Mittag-Leffler function, which is a cornerstone of these models.

Quantum mechanics, statistical physics, and cosmology are some of the theoretical physics fields that have made use of the function. As an example, it has been used to characterise fractional Schrödinger equation-governed quantum systems that exhibit memory effects. The use of the function in cosmological models of dark energy and cosmic development stems from its capacity to characterise power-law behaviours and rapid expansion. The importance of this instrument in comprehending basic natural processes is shown by its many uses.

New research and innovation opportunities are being revealed by the continuous investigation of the Mittag-Leffler function and its generalisations. Its incorporation into existing frameworks for computing and its use in new areas like data science and machine learning bode well for its future development. The flexibility and analytical depth of the Mittag-Leffler function will guarantee its persistence in tackling the intricacies of contemporary systems, even when new scientific and technical difficulties emerge.

The Mittag-Leffler function is a very useful mathematical tool that has many uses in many different branches of science. It has become an indispensable tool in modern mathematics and practical research due to its capacity to generalise classical functions, describe complicated dynamics, and provide analytical solutions to fractional systems. Researchers and practitioners rely on the function because to its mathematical rigour and variety, which allows it to describe anomalous diffusion, viscoelasticity, fractional-order control, and quantum events. It is safe to say that the Mittag-Leffler function will continue to play a pivotal role in defining the trajectory of scientific and technological progress as theory and computing progress.

CONCLUSION

The Mittag-Leffler function, introduced by Gotta Mittag-Leffler in the early twentieth century, is an important instrument in mathematical analysis and applied sciences. It extends the exponential function and interacts with fractional calculus for the first time, making it useful for characterising phenomena including memory effects, anomalous diffusion, and non-linear dynamics. Because of its versatility and rich analytical features, the function has drawn interest in domains such as physics, engineering, control theory, and computer mathematics.

The Mittag-Leffler function is remarkable for its ability to link fractional and classical systems, laying the groundwork for solving fractional differential equations. It can handle a broad range of parameter values, exhibiting its versatility in a number of fields, including the study of anomalous diffusion and non-exponential relaxation in dielectrics.

The Mittag-Leffler function is also employed in materials research to characterise viscoelastic behaviour and non-exponential relaxation in dielectric materials. In engineering and control theory, fractional-order systems often employ the Mittag-Leffler function to simulate system dynamics with memory or hereditary properties. Fractional-order controllers outperform ordinary controllers when dealing with complicated systems, boosting stability and performance.

The Mittag-Leffler function's exceptional mathematical properties contribute to its adaptability, such as completeness, asymptotic properties, convergence for all complex inputs, and integral form. These characteristics make it especially well-suited for dealing with issues involving complex systems, notably in the area of fractional calculus.

The Mittag-Leffler function is a flexible mathematical design that has become an invaluable tool for solving complicated issues in a variety of scientific domains. It generalises the exponential function and has a strong connection to fractional calculus. The function has shown adaptability and utility throughout its history, from its origins in pure mathematics to its many applications in the practical sciences. Mathematical theory and computational techniques have also progressed in response to this function, making them more suited to dealing with complex systems and constantly changing environments.

The Mittag-Leffler function is helpful because it can describe processes that operate in an unusual or non-exponential manner. Classical functions, such as the exponential function, often operate under idealised conditions that may not accurately reflect the complexities encountered in real systems. The Mittag-Leffler function addresses this requirement by enabling tuning parameters to characterise a wide range of behaviours, including memory effects, fractional dynamics, and power-law distributions. It is especially helpful for modelling anomalous diffusion, which defies normal diffusion equations, as well as simulating memory-based systems in viscoelasticity.

Finally, the Mittag-Leffler function exemplifies mathematical creativity's incredible ability to solve complicated problems. As science and technology advance, the Mittag-Leffler function will surely remain an important component in mathematical research and application, driving advancements in our comprehension of the intricacies of our universe and our capacity to articulate them.

The Mittag-Leffler function is a critical mathematical tool in control theory and physical sciences, enabling precise modelling and control of dynamic systems. Its generalisability assures stability and convergence in analytical and numerical applications, making it an effective tool for problem solving in computer science and applied mathematics. The function is most typically employed in fractional differential equations, which extend classical differential equations to incorporate derivatives with non-integer orders.

The Mittag-Leffler function's numerical computing has been investigated for its significance in real-world applications, with advances such as asymptotic approximations, numerical integration techniques, and specialised algorithms allowing for rapid and accurate calculations. The function's scope has expanded to incorporate increasingly complicated systems such enzyme processes, gene expression, and neural network dynamics. It is also used to market dynamics, risk analysis, and option pricing in economics and finance.

The Mittag-Leffler function is also used in quantum mechanics, statistical physics, and cosmology to describe fractional Schrödinger equation-governed quantum systems that exhibit memory effects and power-law behaviour. Its adaptability and analytical depth make it an invaluable resource in a variety of domains, including data science and machine learning.

To summarise, the Mittag-Leffler function is an important mathematical tool in control theory, physical sciences, and engineering due to its capacity to generalise classical functions, explain complicated dynamics, and give analytical solutions to fractional systems.

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