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A Study of Moving Mesh Generation with Mathematical Adaption

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ABSTRACT

In this analysis we will focus on the refining method and present a way to transfer the mesh nodes. Typically it is possible to use the r-refining method to evaluate either the mesh point position or the pace. The mesh can be translated from a theoretical background to the physical space in which the PDE is to be solved and then the rate of change in the time of this mapping can be considered as the speed of the mesh. Every computing node thus has a specific speed, with which it travels, from which the mesh can be progressed in time. This speed can be interpreted as a function of the physical space variable since there is an implicit mapping between machine- and physical areas. Therefore, we need a method to produce this speed and in this step a monitor function is used to create it. This will help to monitor the relative density and hence the degree of mesh adaptation of the mesh points in the physical domain. The way we produce these mesh speeds depends on preserving the Monitor Function Concept in time, which can be used to induce a mesh motion. An Eulerian conservation law can be extracted from this concept to ensure that the mesh speed is obtainable from the display. This conservation legislation for the mesh motions is used to achieve a unique mesh speed in combination with a curl condition that determines the rotating properties of the mesh. In conjunction with the curl conditions, the conservation law then shows that the mesh-speed potentials can be derived from an elliptic equation.

KEYWORD

Moving Mesh, Mathematical Adaption, conservation law, elliptic equation

INTRODUCTION

Moving mesh generation and adaptation techniques to generate irregular grid systems that can solve both normal and partial differential equations. There are a wide range of methods developed that typically fall into one of two major categories. These categories are referred to as methods based on speed or position. By adding the speeds of the measurement nodes in the mesh, methods based on the speed are derived. In the case of a mapping from a reference mesh the positions are used to search the position of the computational nodes.

Most methods used to create an irregular mesh are designed as a transformation from a computer domain to a physical domain in which the specific problem is solved. In this paper, x and ξ denote physical coordinates and computational

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coordinates, Ω and Ωc,which are defined respectively in the areas of both gas and tunnel. Then we suppose that there exists a one-to-one transformation denoted by

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t),$$

that takes points into physical space in the computer space at a certain time. This is shown in Figure 1. The goal of an abnormal mesh is to

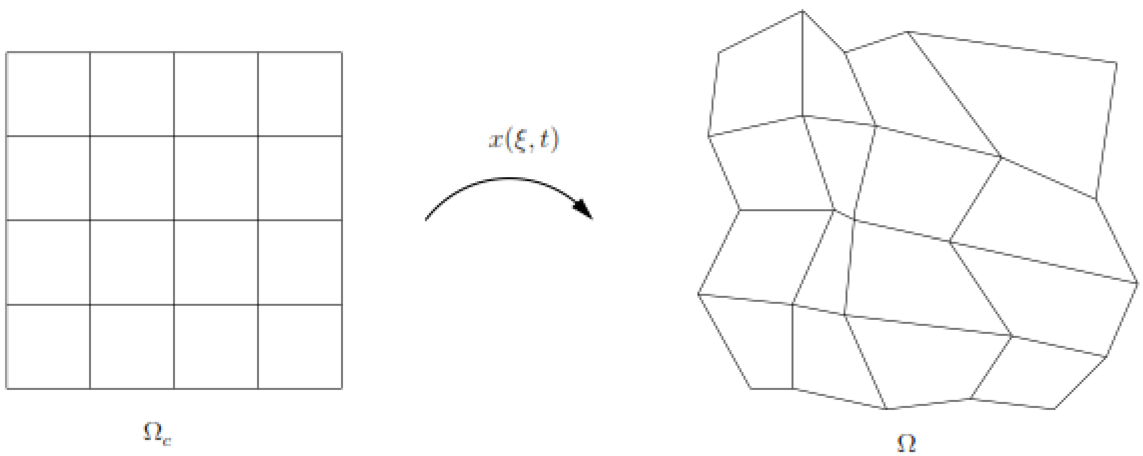

Figure 1: Figure showing the mapping x(ξ,t) from the background computational mesh Ωc to the physical mesh Ω.

Check this mapping to achieve the desired mesh points clustering and computer cell alignment. We now look at how this mapping can be created. Equidistribution is one of the most common concepts used to create a non-uniform mesh. De Boor [16] implemented the equidistribution principle to achieve a discreet approximation of the function on a non-uniform mesh and selection of mesh points so the integral part of a certain measurement is equalised over each mesh computer cell. This measure is specified by the consumer and is called the monitor function. In order to represent the numerical solution or function approximation on the grid, the monitor function is chosen. The solution and/or its derivatives may be positive and dependent upon it. A general shape of the monitor works in multiple dimensions is given by.

$$M = M(\mathbf{x}, t, u, u_x, u_y, \ldots).$$

Where u is the approximate function. This function may be a function or the solution for a differential equation. We are now going to define the concept of equidistribution in one spatial aspect and then look at the extensions. The domains ̈Ω and Ωc were the unit interval [1, 1] without loss of generality [0, 1] The domains in real space, so x, ξ ∈ [0, 1].

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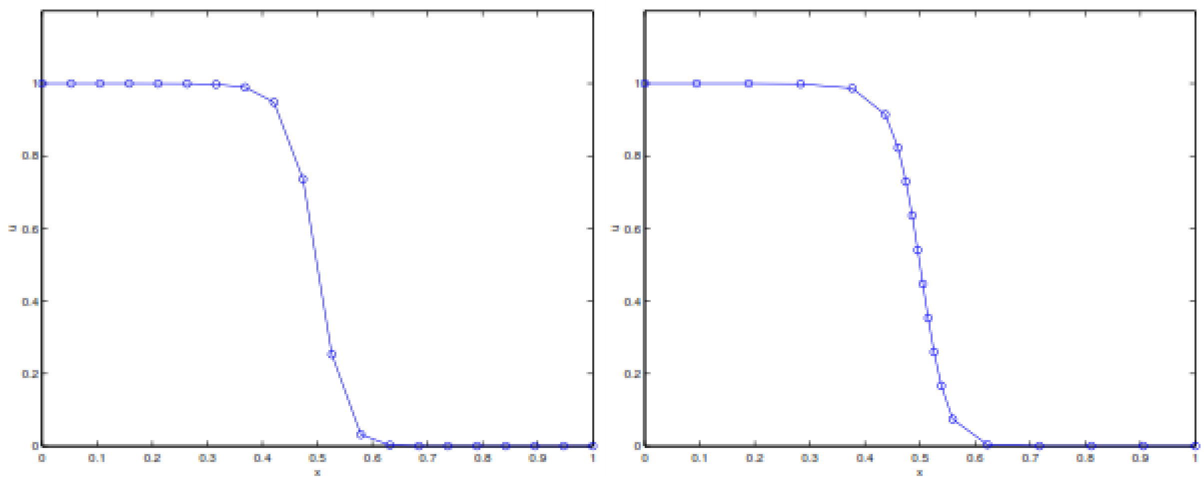

Figure 2: Figures illustrating the equidistribution principle. (Left) a function represented on equally spaced mesh (right) the same function represented using equal arclength. Both graphs are computed with 20 computational nodes

The mapping conditions x(0,t) = 0 and x(1,t) = 1 are also limited. Then there is the continuous form of the theory of equidistribution as White [99]. is given as

$$\int_0^{x(\xi,t)} M\left(x\left(\xi,t\right),t\right)\,\mathrm{d}x = \xi \int_0^1 M\left(x\left(\xi,t\right),t\right)\,\mathrm{d}x, \quad \forall t. \tag{1}$$

We are now offering the equidistribution concept a clear example. Take the feature into account.

$$u(x) = \frac{1}{2} - \frac{1}{2}\tanh\left(20\left(x^2 - \frac{1}{4}\right)\right)$$

Interval x [0, 1] is defined. With only a small number of mesh points, we want this feature discreetly to be portrayed as

$$0 \equiv x_0 < x_1 < \cdots < x_{N-1} < x_N \equiv 1.$$

The feature u depicted in two separate meshes is illustrated in Figure 2. On a mesh with the same length, the relation figure shows the function: xi − xi−1 = xi+1 − xi for I = 1, . N −1. . It should be noted that this mesh can be regarded as being generated from the equidistribution principle (1). Alternatively, the right 8 figure shows the same function, but this time the arc-length monitor function is depicted on a mesh built from the equal allocation principle

$$M = \sqrt{1 + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2}.$$

This choice of monitoring function leads to a more mesh points clustering in and around regions of large function variance and seems to give the function a 'better' discreet approximation.

The equidistribution theory of (1) may be distinguished following Huang, Ren and Russell[49]. with respect to ξ to obtain.

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$$M\left(x\left(\xi,t\right),t\right)\frac{\partial}{\partial\xi}x\left(\xi,t\right)=\theta(t), \qquad (2)$$

where θ(t) = R 1 0 M (x (ξ,t),t) dx. White numerically solved the time-independent version of equation (2) in [99] in order to generate an adaptive grid for solving two-point border-value problems. It gives twice the distinction between the concept of equidistribution and that of ξ gives

$$\frac{\partial}{\partial\xi}\left(M\left(x\left(\xi,t\right),t\right)\frac{\partial}{\partial\xi}x\left(\xi,t\right)\right)=0. \qquad (3)$$

Equations (2) and (3) are defined as the Quasi-Static Concepts of Equidistribution, which does not include details on the movement of computational nodes. Baines has numerically resolved the time-independent version of the equation (3) by using an iterative method, while concerns regarding the stability of such solvers have been asked. As the monitor function is usually x, equation (3) is a non-linear equation of the mesh location so that the equation is iteratively solved,

$$\frac{\partial}{\partial\xi}\left(M(x^p)\frac{\partial x^{p+1}}{\partial\xi}\right)=0,$$

Where p is the counter for iteration. Dismissing this balance results in a tridiagonal system that can be resolved with an iteration from Jacobi. Where a ninth equidistributed mesh is generated to provide a good approximation of a specific feature, monitor values are available and this is typically an iterative operation. However, it is more computationally effective to use a mesh method when a mesh is used to solve a differential equation where the mesh and solution are alternatively modified. Interpolation can be used to move the solution from the old mesh into the new mesh when a new mesh is created. Flaherty et al derive a moveable mesh equation by distinguishing between equal distribution (1) and time to be obtained.

Moving Mesh Methods in Higher Dimensions

1 Grid Generation

This section begins with a summary of several common methods of multidimensional static mesh generation. We will then explore how these techniques of static mesh generation can be generalised to deal with time-based problems. One of the earliest ways to create a mesh is probably Winslow in multi-dimensions. The ideas behind this method set several techniques to be followed for mesh generation. The core concept behind Winslow's approach is to formulate the problem of mesh generation as a possible problem where the mesh lines are compatible with equipment. In order to describe the mesh, the equipment lines can be used. (There is only an appendix to the problem of mesh generation to which deals primarily with the solution of quasi linear PDEs on triangular meshes) The approach is as follows. Let two sets of equipotentials defined as $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ satisfy the Laplace equations

$$\nabla^2\xi = 0 \qquad (4)$$

$$\nabla^2\eta = 0 \qquad (5)$$

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in some region Ω. The solution to equations (4) and (5) provides equipment ξ = constant and η= constant for a mesh to be formed from by taking these lines into account. The equipment = constant. By reversing mappings, the desired mesh can be obtained numerically to give x = x(ξ, η) and y = y(ξ, η). Using the Jacobian determinant J = xξ yη − xη yξ equations (4) and (5)

2 Links with Equidistribution

We have considered in one dimension various moving mesh equations derived from the theory of equidistribution. Also, the Euler-Lagrange equation can be written in one dimensions (1) for reducing functionality.

$$I[\boldsymbol{\xi}] = \int_\Omega \frac{1}{M(\mathbf{x})} \left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}\right)^2 \mathrm{d}x. \qquad (6)$$

However, this reduction theory is not specified in a multifaceted way (6) as it would require. that we have

$$\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}} = M(\mathbf{x}),$$

Where now M(x) is a n matrix that regulates the different mapping characteristics. It is difficult to solve this method, as it is too much. Thus, the Knupp theory is used to decide how the mapping is to be followed

$$I[\boldsymbol{\xi}] = \int_\Omega \left|\left|\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}} - M(\mathbf{x})\right|\right|_F^2 \mathrm{d}\Omega$$

where || · ||F is the Frobenius norm of the matrix

Baines in [7] considered a natural generalisation of the one dimensional equidistribution principle. The equation

$$\nabla_\zeta \cdot (M(n)\, \nabla_\zeta n) = 0, \qquad (7)$$

Where n is a coordinate along the gradient direction of the solution, the mesh points are moved ∇u, and ζ = (ξ, η)). This equation for the mesh reduces the perpendicular concept of one-dimensional equidistribution. Baines also states that the useful mesh adaptive technique is either substituted by n in the equation (7) by x or y

$$\nabla_\zeta \cdot (M\, \nabla_\zeta x) = \nabla_\zeta \cdot (M\, \nabla_\zeta y) = 0 \qquad (8)$$

is obtained. These equations are used to construct mesh adjusted to functions with either Dirichlet or Neumann boundary conditions using the arc-length monitor function. The mesh and solution are resolved on an iterative basis, and it is found that the resulting mesh does not distribute the monitor's function strictly equitably. The system, however, is found in broad M regions to be cluster points and thus produces fairly convincing meshes. All approaches in two dimensions detailed so far only for the generation of one adaptive mesh have been defined. However, an adaptive mesh is required every step when time-dependent problems are resolved. This is achieved in two major ways. First, the static mesh generator is simply used at each step, and a number of meshes therefore are generated. Unfortunately, this process is not necessarily a successful process, and the mesh can sometimes shift very quickly.

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3 Links with Fluid Dynamic

The classic approach in Lagrangia is a method for moving the machine mesh with fluid problems where the speed of the nodes is supposed to be equal to the real fluid speed. We therefore have to give the mesh speed.

$$\dot{\mathbf{x}} = \mathbf{v},$$

Where v is the fluid speed. There are many benefits to this approach, but there are unfortunately some significant downsides. The Lagrangian approach maintains a good resolution of solution compressions and extensions and also maintains fluid multi-material interfaces. However, the Lagrangian mesh can easily be singular for compressible flow calculations in greater than one spatial dimension. Meshes are gradually distorted as vorticity and shear come in the flow and in the final period can become singular. The mesh is kept in time in the Eulerian system, and the fluid moved through the mesh. Therefore, the mesh speed is x = 0. = 0. Because of this, the meshes of Euler are not tangled and become distinct. But one issue with numerical solutions on Eulerian meshes is their excessive dissemination as well as material interfaces which are hard to retain. The goal of Hirt, Amsden and cook (ALE) methods is to try to incorporate the best of the Eulerian and Lagrangian methods. Their aim is to combine the best components of the ALE methods. The key theory behind ALE is that neither Eulerian nor Lagrangian mesh movement has to be limited and that a solution technique can be arbitrarily chosen to improve accuracy and stability. The partial difference equation is fixed in a moving frame of reference after the mesh movement has been chosen. The integral protection of mass equation, for example

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho \,\mathrm{d}\Omega + \int_{\Omega(t)} \nabla \cdot \rho\,\mathbf{v}\,\mathrm{d}\Omega = 0$$

in a general reference frame moving with velocity $\dot{\mathbf{x}}$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho \,\mathrm{d}\Omega + \int_{\Omega(t)} \nabla \cdot \rho\,(\mathbf{v} - \dot{\mathbf{x}})\;\mathrm{d}\Omega = 0.$$

Where ¨is the fluid density to be fixed. To solve a number of applications, ALE processes were successfully employed.

4 Moving Finite Elements

The Moving Finite Elements (MFE), which Miller and Miller developed, is another velocity-based process. Initially, the MFE approach was used to approximate time-based partial equations of the form

$$u_t = \mathcal{L}u$$

that had steep frontal movement. The key principle behind MFE was that the residual L2 norm of the differential PDE shape was minimised. You may write this in the form

$$\min_{\dot{\mathrm{X}},\frac{\mathrm{D}U}{\mathrm{D}t}} \int_{\Omega} \left(\frac{\mathrm{D}U}{\mathrm{D}t} - \dot{\mathrm{X}} \cdot \nabla U - \mathcal{L}U \right)^2 w \,\mathrm{d}\Omega, \tag{9}$$

And the regular functions of the nodal finite element base here. The weight function in (9) was taken from MFE in the original version as w version w ≡ 1. But this weight function may also be chosen to depend on a solution, so it can depend on a solution u or its by-products. The weight feature is widely used $w \equiv \frac{1}{\sqrt{(1+|\nabla U|^2)}}$. This weight selection results in weighted gradient moving final elements (GWMFE). MFE reduction is carried out via the derivation of the usual equations.

$$\int_{\Omega} \left(\frac{\mathrm{D}U}{\mathrm{D}t} - \dot{\mathrm{X}} \cdot \nabla U - \mathcal{L}U \right) w \, \mathrm{d}\Omega = 0 \quad (10)$$

$$\int_{\Omega} \left(\frac{\mathrm{D}U}{\mathrm{D}t} - \dot{\mathrm{X}} \cdot \nabla U - \mathcal{L}U \right) \nabla U \, w \, \mathrm{d}\Omega = 0 \quad (11)$$

for the minimisation over $\frac{\mathrm{D}U}{\mathrm{D}t}$ and $X^{\cdot}$ respectively. In fact, weak forms of PDE that are resolved are easily seen. The mesh speed obtained by MFE can be approximately laagrangian in certain circumstances [6]. The MFE Approach is primarily designed to minimise the weight of the residual L2 norm of the discerning PDE that is being resolved by the mesh movement. One of the problems with the MFE approach is that it is possible to construct unique matrices from [10 and 11]. Therefore, in order to produce acceptable results, some problems involve careful regularisation of the procedure. This method of regularisation typically involves applying penalties to the remaining norm and thus the standard equations in order to prevent them from being special.

In most situations the techniques mentioned in this study so far will lead to good results in partial differential equations on moving meshes for the numerical solution. However, these methods can lead to poor results and sometimes spurious solutions for certain problems unless additional care is taken. Recent interest was therefore given to geometrical methods designed to inherit some or all of the device structure.

GEOMETRIC INTEGRATION

The aim of geometric integration is to develop discreet approximations to continuous differential equations systems that retain key system features. An excellent analysis paper on geometric integration and its applications is available. The use of moving mesh procedures for the solution of partial difference equations with invariant behaviour and self-similar solutions has been researched extensively. The Porous Medium Equation is one unique PDE solved using this kind of geometric inclusion approach (PME). In Budd and Collins create a moveable discrete mesh of the SME. The purpose of the work is to develop discreet solutions to SMEs that have the same asymptotic output as solutions to the underlying continuous problem. The numerical system is achieved by ensuring that the properties of the continuous equation are preserved. The scheme has the same preservation properties and the same invariants, in particular. By using a finite-differential MOL method, the PME system is semi-discreted and then transferred to ensure discreet mass preservation in the numerical solution. The resulting discrete system then shows that the invariants in the continuous problem are discrete and identical. They also show that the discretization error is time-independent, in contrast to non-invariant moving mesh techniques, where errours expand with the expansion of the spatial domain and the increasing spatial phase scale. In Budd et al., the SME is again considered.

The SME is also solved with a moveable mesh technique by Budd and Piggott again. They argue that if PDEs are used to show scale invariant behaviour, the monitor function of the moving mesh MMPDE6 equation should in some way be invariant. They therefore use the so-called mass surveillance feature.

$$M = u.$$

It also ensures that the mesh moves discreetly to protect the solution's mass. Budd, Huang and Russell consider in the adaptive PDE solution

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$$u_t = u_{xx} + f(u), \qquad (12)$$

with particular reference to the cases when $f(u) = u^p$ where p > 1 is a parameter and $f(u) = e^u$. It can be shown that when the initial condition $u_0(x)$ for The PDE solution will blow-up in finite time these issues are "sufficiently big." This form of conduct can be hard to estimate with conventional fixed grid methods since the precise nature of the blow-out function of the solution can deteriorate considerably when the spatial size of the singularity is lowered to below the fixed spatial phase size. Budd and others use MMPDE6 and the monitor feature to build a moveable mesh.

$$M(u) = u^{p-1}$$

for the case when. $f(u) = u^p$ It is chosen because the mesh equation scale remains invariant to the scale of the underlying continuous problem. The Budd and the Cluster mesh points in the blow-up area can be accurately captured by a moving mesh system for resolving problems of blow-up.

Budd et al. in apply similar methods of the non-linear Schr Âoderinger equation to the blow-up problem. The Schr¨odinger nonlinear equation is stated.

$$i\frac{\partial u}{\partial t} + \nabla^2 u + |u|^2 u = 0.$$

The result was a moving mesh method used to define a suitable monitor function by using scale invariant techniques. The invariant scale monitor functions are considered $M = |u|^2 \text{ and } M = \sqrt{\alpha |u|^4 + \beta |u_r|^2}$ Where r are constants, α and β are radial co-ordinates. MMPDE6 was used then to transfer the mesh. The non-linear blow-up problem is considered and the moving mesh calculations give precise predictions of the blow-up area.

In Blake used a number of parabolic partial differential equations by using geometric techniques. The moving mesh process used is mainly designed for the SME solution to create a system that locally preserves the mass of the solution in a distinct way. The moving mesh equation is discreetly combined with a final difference system by a backwards differentiation formula (BDF). The Mass Monitor M = u was the first monitor feature to be used because of the monitor's scale invariance. It was found that this monitoring feature produces good results for the PME when the gradient of solution was not too high. However, the method indicates that steep motion fronts are not adequately resolved, so the monitor feature was altered to try to enhance the method accuracy.

CONCLUSION

We conclude this thesis by considering some other possible areas of further research. There are a variety of avenues for further research using the moving mesh method that has been used in this thesis. One could solve many other partial deferential equations or apply the method to the PDEs already solved in other contexts, as well as devising more complicated test problems such as the Mach reaction test problem for the compressible Euler equations. Alternatively, we could pursue the design of deferent monitor functions which could result in a more accurate and robust method. Error estimates may be used as monitors if available, or monitor functions may be designed to bring out particular features of the solution. For example, it may be advantageous for the solution of the compressible Euler equations to have a monitor function based on the curvature of the density, or some other variable, to move mesh points into shock and contact wave regions. This type of monitor function may also be combined with the already utilised gradient monitor function.

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