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A Study of Mass Conserving Function Solution of Differential Equation

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ABSTRACT

In this study we consider the adaptive solution of time-dependent partial differential equations using a moving mesh technique. A moving mesh method is developed utilising monitor functions to drive the mesh motion and is based on equidistribution ideas. The method is derived in multidimensions from a conservation of monitor function principle and from this initial principle a conservation law for the monitor function is derived which generates the velocity of the mesh. In dimensions higher than one the mesh velocity is underdetermined for this conservation law, therefore the curl of the velocity field is prescribed to obtain a unique velocity field. The conservation law together with the curl condition is combined to produce an elliptic equation for a mesh velocity potential, from which the mesh velocity can be obtained. The moving mesh equations are solved for the mesh velocity using standard linear finite elements and then a new mesh is constructed by integrating the mesh velocity forward in time using finite differences. The moving mesh method is applied to the adaptive solution of parabolic and hyperbolic equations. The Partial Differential Equations (PDE) with a moving boundary problem is solved taking advantage of scale invariant properties. For this problem the solution to the PDE is obtained through the conservation of monitor function for two different monitor functions; a mass and a gradient monitor function.

KEYWORDS

Mass Conserving Function, Differential Equation, moving mesh technique, Partial Differential Equations, monitor function, PDE

INTRODUCTION

The Partial Differential Equations (PDE) and determining some of its properties, we will begin this chapter. The PME is a parabola-like, non-linear partial differential equation that comes mainly from the study of ideal gases flowing through a porous medium. It also inevitably emerges as a model for many physical phenomena, such as the swarming of different species of insects, the dispersal of small, undergravity fluids and radiative warming.

The porous medium equation (1) in many dimensions is given as

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$$u_t = \nabla \cdot (u^m \nabla u)\,, \qquad (1)$$

Where u = u(x,t), the scalar function is a constant that is generally taken as a positive integer depending upon the x and t, spatial and temporal variables and the m. In Darcy's law on speed to pressure gradient, the poreous medium equation can be derived from considering gas diffusion through a poreous medium. Density (u), pressure (p) and speed are the variables characterised by the flow of the gas (v). The maintenance of the mass equation is believed to occur by

$$\rho\, u_t + \nabla \cdot (u\, \mathbf{v}) = 0, \qquad (2)$$

where ρ is the constant porosity of the medium, and Darcy's law which is given as

$$\mu\, \mathbf{v} = -\kappa\, \nabla p. \qquad (3)$$

The law of Darcy is an empirical law for porous media flow dynamics. In this case μ is the gas viscosity and μ is the medium permeability, both meant to be continuous. The gas is also called optimal, so that the density pressure is associated by

$$p = p_0\, u^\gamma, \qquad (4)$$

where p0 is the reference pressure and γ is the ratio of specific heats for the gas. If equations (3) and (4) are substituted into equation (2) the equation

$$u_t = c\, \nabla \cdot (u^\gamma \nabla u)\,,$$

is obtained, where c is a constant given as

$$c = \frac{\kappa\, p_0\, \gamma}{\mu\, \rho}.$$

The constant c can be scaled off and if this is accomplished and m is called μ then we get the PME (1). The analytical properties of SME have been extensively studied. In view of the initial condition u(x, 0), SME solutions maintain two essential quantities: mass and centre of mass. The full mass of a solution is met in view of the SME's solution which fulfills the conditions of both u → 0 and u m∇u → 0 as x → ∞, complete the solution

$$\frac{\mathrm{d}}{\mathrm{d}t}\int u\, \mathrm{d}\Omega = \int u_t\, \mathrm{d}\Omega = \int \nabla \cdot (u^m\, \nabla u)\ \mathrm{d}\Omega = \oint u^m \nabla u \cdot \mathrm{d}\boldsymbol{\Gamma} \to 0 \quad \text{as } \mathbf{x} \to \infty. \qquad (5)$$

Thus the PME conserves mass. Similarly, if the centre of mass is given by

$$\bar{\mathbf{x}} = \int \mathbf{x}\, u\, \mathrm{d}\Omega$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{x}\,u\,\mathrm{d}\Omega &= \int \mathbf{x}\,u_t\,\mathrm{d}\Omega = \int \mathbf{x}\,\nabla\cdot\left(u^m\,\nabla u\right)\,\mathrm{d}\Omega. \\ &= \oint \mathbf{x}\,u^m\,\nabla u\cdot \mathrm{d}\mathbf{\Gamma} - \frac{1}{m+1}\int\left(\nabla\cdot\mathbf{x}\right)\,\nabla\left(u^{m+1}\right)\,\mathrm{d}\Omega \\ &= \oint \mathbf{x}\,u^m\,\nabla u\cdot \mathrm{d}\mathbf{\Gamma} - \frac{1}{m+1}\,\nabla\cdot\mathbf{x}\oint\left(u^{m+1}\right)\,\mathrm{d}\mathbf{\Gamma}\to 0 \quad \text{as } \mathbf{x}\to\infty.\end{aligned}$$

Thus the PME also conserves the centre of mass of a given solution.

The analysis of scale invariance results in special solutions for the SME that preserve mass and centre of mass and are referred to as solutions for similarity. In the following section we will describe the invariance of the scale and how similar solutions for the SME can therefore be created.

Scale Invariance and Similarity Solutions

Scale invariance can be described as follows. Given the set of variables (u, x,t) which satisfy the PDE under consideration, introduce a mapping to a new system (u, ˆ xˆ, tˆ) given by the scaling transformation

$$\hat{u}\to\lambda^{\alpha}\,u, \quad \hat{\mathbf{x}}\to\lambda^{\beta}\,\mathbf{x} \quad \hat{t}\to\lambda\,t \qquad (6)$$

where α and β are exponents to be found and the constant λ is arbitrary. Then the system is said to be scale invariant if the mapping from (u, x,t) to (u, ˆ xˆ, tˆ) leaves the PDE unchanged

We will now seek the transformation of the form (6) which leaves the PME, in radial co-ordinates, given

$$u_t = \frac{1}{r^{d-1}}\left(r^{d-1}\,u^m\,u_r\right)_r, \qquad (7)$$

where r is the radial co-ordinate and d is the number of spatial dimensions, invariant. Carrying out a change of variables of the form (6) the radial porous medium equation (7) becomes

$$\lambda^{1-\alpha}\,\hat{u}_{\hat{t}} = \lambda^{2\beta-(m+1)\,\alpha}\,\frac{1}{\hat{r}^{d-1}}\left(\hat{r}^{d-1}\hat{u}^m\,\hat{u}_{\hat{r}}\right)_{\hat{r}}.$$

Therefore the radial PME will be scale invariant under the transformation (u, r,t) → (u, ˆ rˆ,tˆ) provided that

$$1-\alpha = 2\beta-(m+1)\,\alpha,$$

Or,

$$m\,\alpha - 2\beta + 1 = 0 \qquad (8)$$

holds. We shall also require that the conservation of mass principle should hold in the transformed co-ordinates. The conservation of mass principle in radial co-ordinates in d dimensions is given by

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$$\int_0^\infty u\, r^{d-1}\, \mathrm{d}r = \text{constant} \quad \forall t.$$

Substituting the change of variables into the above conservation principle we find that

$$\lambda^{-\alpha-d\beta} \int_0^\infty \hat{u}\, \hat{r}^{d-1}\, \mathrm{d}\hat{r} = \text{constant} \quad \forall t$$

Provided that

$$\alpha + d\,\beta = 0. \qquad (9)$$

Therefore using the algebraic equations (8) and (9) we find that the radial porous medium equation and its conservation property are invariant under the transformation

$$\hat{u} \to \lambda^{\alpha} u, \quad \hat{r} \to \lambda^{\beta} r \quad \hat{t} \to \lambda t \qquad (10)$$

If,

$$\alpha = -\frac{d}{d\,m+2} \quad \text{and} \quad \beta = \frac{1}{d\,m+2}. \qquad (11)$$

However, we note that the preservation of the mass property core is not invariant in this scale.

We are now describing solutions that are self-similar to SMEs. Any SME solution shall also be invariant under (10) and (11) transformations and thus satisfy.

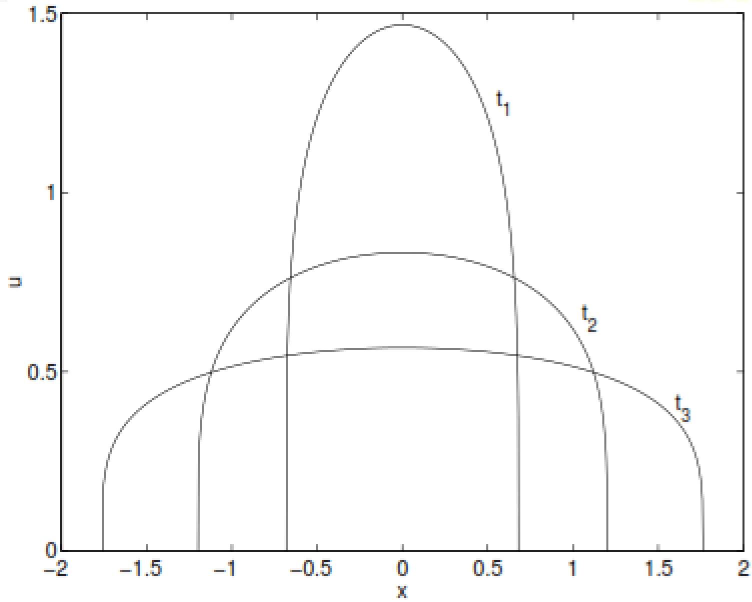

Figure 1: Figure depicting the radial similarity solution in one spatial dimension (d = 1) at three different times t1 < t2 < t3 with m = 4

A MASS CONSERVING MONITOR FUNCTION

In one spatial dimension (d = 1), it is considered and solved by means of the moving mesh approach which was derived. The goal is to derive a method that produces numerical solutions that have the same asymptotic behaviour. We hope that this is achieved by designing a scheme that retains and scales the same as SMBs.

Substantial forms of the equations (12), (13) were also derived and (14). Therefore the distributed protection of the monitor function theory is distributed in one dimension (15).

$$\int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i M \, \mathrm{d}x = \theta_i \qquad (12)$$

and the mesh velocity potential equation

$$\int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i \frac{\partial}{\partial x}\left(M \frac{\partial \phi}{\partial x}\right) \mathrm{d}x = -\int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i \frac{\partial M}{\partial t} \mathrm{d}x$$

for $i = 1, \ldots, N-1$. (13)

For our issue we now need to pick an acceptable monitoring feature. The initial choice is M = u, due to the fact that the mass of the solution for the PME remains unchanged and also since (15) is then invariant in size. We hope to create a method from this monitor feature that preserves mass discreetly.

Therefore the equation (15) for interior nodes becomes with this monitor function.

$$\int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i U \, \mathrm{d}x = \theta_i, \qquad (15)$$

When θi the initial mesh and solution is calculated as a matter of course. If we now increase approximation to the finite element in terms of critical functions, we get the matrix system

$$A(\underset{\sim}{X}) \, \underset{\sim}{U} = \underset{\sim}{\theta}, \qquad (16)$$

Where *A* is the usual finite element mass matrix with entries given by

$$A_{ij} = \int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i \, w_j \, \mathrm{d}x.$$

We have made it clear that U = 0 is at the end of the domain. The velocity potential equation (16) can be achieved with the same monitor function

$$\int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i \frac{\partial}{\partial x}\left(u \frac{\partial \phi}{\partial x}\right) \mathrm{d}x = -\int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i \frac{\partial u}{\partial t} \mathrm{d}x.$$

A weak form of the PME can be substituted into the right hand side of the velocity potential equation above to give

$$\int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i \frac{\partial}{\partial x}\left(u\frac{\partial \phi}{\partial x}\right) \mathrm{d}x = -\int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i \frac{\partial}{\partial x}\left(u^m\frac{\partial u}{\partial x}\right) \mathrm{d}x.$$

To obtain a weak form of the velocity potential equation we integrate by parts to give

$$\left[w_i\, u\frac{\partial \phi}{\partial x}\right]_{x_{i-1}(t)}^{x_{i+1}(t)} - \int_{x_{i-1}(t)}^{x_{i+1}(t)} \frac{\partial w_i}{\partial x}\, u\frac{\partial \phi}{\partial x} \mathrm{d}x = -\left[w_i\, u^m\frac{\partial u}{\partial x}\right]_{x_{i-1}(t)}^{x_{i+1}(t)} + \int_{x_{i-1}(t)}^{x_{i+1}(t)} \frac{\partial w_i}{\partial x}\, u^m\frac{\partial u}{\partial x} \mathrm{d}x.$$

The first term vanishes for interior nodes since the basis function w_i is equal to zero at $x_{i-1}(t)$ and $x_{i+1}(t)$. We will use the boundary conditions that $u = 0$ and $\phi = 0$ at the boundary of the domain to produce

$$-\int_{x_{i-1}(t)}^{x_{i+1}(t)} \frac{\partial w_i}{\partial x}\, u\frac{\partial \phi}{\partial x} \mathrm{d}x = \int_{x_{i-1}(t)}^{x_{i+1}(t)} \frac{\partial w_i}{\partial x}\, u^m\frac{\partial u}{\partial x} \mathrm{d}x$$

for all nodes. Once the finite element approximations have been substituted in this equation and expanded in terms of the basic functions we obtain a weighted stiffness matrix system for the mesh velocity potential Φ and has the form

$$K\,\underset{\sim}{\Phi} = \underset{\sim}{f}$$

where the entries of the weighted stiffness matrix are given by

$$K_{ij} = -\int_{x_{i-1}(t)}^{x_{i+1}(t)} \frac{\partial w_i}{\partial x}\, U\, \frac{\partial w_j}{\partial x} \mathrm{d}x.$$

The mesh velocity can then be recovered by solving the mass matrix system which results from enforcing

$$\int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i\, \dot{X}\, \mathrm{d}x = \int_{x_{i-1}(t)}^{x_{i+1}(t)} w_i\, \frac{\partial \Phi}{\partial x} \mathrm{d}x.$$

leading to

$$A\,\underset{\sim}{\dot{X}} = \underset{\sim}{b}.$$

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The matrices generated by the function Monitor M = u are tridiagonal and can be solved by a direct tridiagonal solver.

The mesh is integrated in time with a time step process for Euler. The technique is stable on condition, but robust otherwise. But if the ODE system is solved using a regular algorithm

$$\dot{\underset{\sim}{X}} = \underset{\sim}{F}(\underset{\sim}{X})$$

given by

$$\frac{\underset{\sim}{X}^{n+1} - \underset{\sim}{X}^{n}}{\Delta t} = \underset{\sim}{F}(\underset{\sim}{X}^{n})$$

We're not going to necessarily obtain an invariant scale system. This is because the local number method truncation error has an underlying length scale and is not independent of time. One way to avoid this problem and to recover a scale approach is to solve the ODE scheme instead.

The outline of the numerical method is:

• Set up the rigidity matrix framework for the mesh velociity potential

• Due to an initial mesh, solution and time phase.

• Recover the mesh velocity X / 1 via the invariant scale forward mesh procedure of Euler time step and time step by mesh

• Get a U solution on the new mesh via the resolution of the equation theory of display preservation.

• Repeat until you hit the desired production time.

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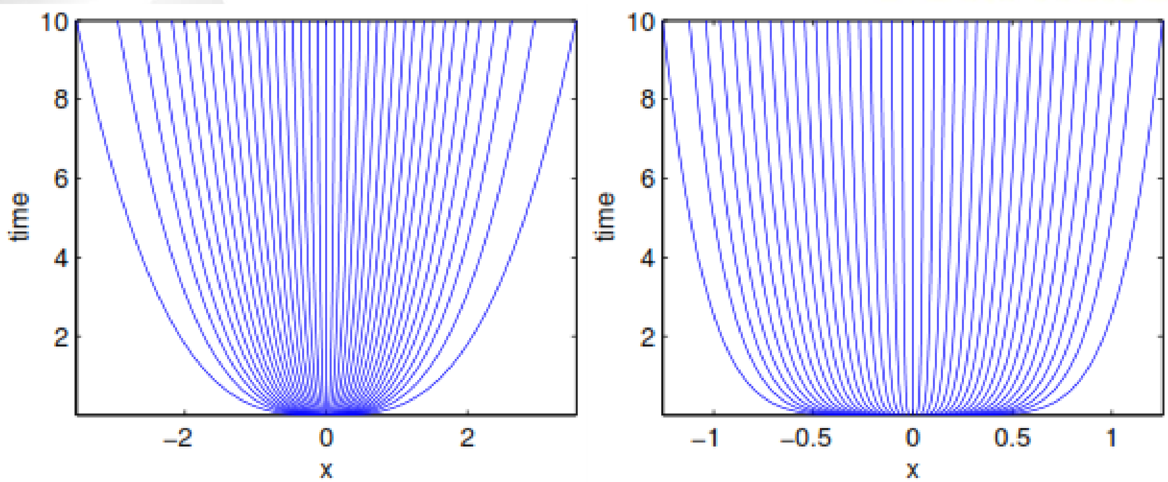

Figure 2: Node trajectories for m = 1 (left) and m = 4 (right).

CONCLUSION

Moving mesh is generated which is highly skewed due to the presence of large amounts of vorticity in the solution then we may want to choose the curl of the Moving mesh velocity in such a way as to minimise the amount of skewness in the Moving mesh. We may also investigate how the solution of the problem being solved depends on the initial Moving mesh chosen. For the two-dimensional results obtained for both the porous medium equation and the Euler equations we used a relatively uniform initial Moving mesh, however we may be able to obtain more accurate results by using an initially adapted Moving mesh. For example, we could use a least-squares _t with adjustable nodes to produce an initial mesh. Alternatively it may be possible to construct a non-uniform mesh from an uniform Moving mesh by using the moving mesh method described in this thesis. This could be done by considering the initial grid generation as a steady state problem and advancing the mesh forward in pseudo-time to produce an adapted Moving mesh.

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