Study of Different Polynomial and Matrix Fractions

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Abstract: ***This article illustrates how polynomials and polynomial matrices can be used to describe linear systems. The focus is put on the transformation to and from the state-space equations, because it is a convenient way to introduce gradually the most important properties of polynomials and polynomial matrices, such as: coprimeness, greatest common divisors, unimodularity, column- and row- reducedness, canonical Hermite or Popov forms.***

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1. INTRODUCTION

The first step when studying and designing a control strategy for a physical system is the development of mathematical equations that describe the system. These equations are obtained by applying various physical laws such as Kirchoff's voltage and current laws (electrical systems) or Newton's law (mechanical systems). The equations that describe the physical system may have different forms.

They may be linear equations, nonlinear equations, integral equations, difference equations, differential equations and so on. Depending on the problem being treated, one type of equation may prove more suitable than others.

The linear equations used to describe linear systems are generally limited either to

- The input-output description, or external description in the frequency domain, where the equations describe the relationship between the system input and system output in the Laplace transform domain (continuous-time systems) or in the z-transform domain (discrete-time systems), or
- The state-variable equation description, or internal description, a set of first-order linear differential equations (continuous-time systems) or difference equations (discrete-time systems).

Prior to 1960, the design of control systems had been mostly carried out by using transfer functions. However, the design had been limited to the single variable, or single-input-single-output (SISO) case. Its extension to the multivariable, or multi-input-multi-output (MIMO) case had not been successful. The state-variable approach was developed in the sixties, and a number of new results were established in the SISO and MIMO cases. At that time, these results were not available in the transfer-function, or polynomial approach, so the interest in this approach was renewed in the seventies. Now most of the results are available both in the state-space and polynomial settings.

The essential difference between the state-space approach and the polynomial approach resides in the practical way control problems are solved. Roughly speaking, the state-space approach heavily relies on the theory of real and complex matrices, whereas the polynomial approach is based on the theory of polynomials and polynomial matrices. For historical reasons, the computer aided control system design packages have been mostly developed in the late eighties and nineties for solving control problems formulated in the state-space approach. Polynomial techniques, generally simpler in concepts, were most notably favored by lecturers teaching the basics of control systems, and the numerical aspects have been left aside. Recent results tend however to counterbalance the trend, and several reliable and efficient numerical tools are now available to solve problems involving polynomials and polynomial matrices. In particular, the Polynomial Toolbox for Matlab is recommended for numerical computations with polynomials and polynomial matrices.

Whereas the notion of the state variable of a linear systems may sometimes sounds somehow artificial, polynomials and polynomial matrices arise naturally when modeling dynamical systems. Polynomial matrices can be found in a variety of applications in science and engineering. Second degree polynomial matrices arise in the control of large flexible space structures, earthquake engineering, the control of mechanical multi-body systems, and stabilization of damped gyroscopic systems, robotics, and vibration control in structural dynamics. For illustration, natural modes and frequencies of a vibrating structure such as the Millennium footbridge over the river

Thames in London are captured by the zeros of a quadratic polynomial matrix. Third degree polynomial matrices are sometimes used in aero-acoustics. In fluid mechanics the study of the spatial stability of the Orr-Sommerfeld equation yields a quartic matrix polynomial.

In this article, we will describe a series of concepts related to polynomial matrices. We will introduce them gradually, as they naturally arise when studying standard transformations to and from the state- space domain.

2. SCALAR SYSTEMS

2.1. RATIONAL TRANSFER FUNCTION

Assuming that the knowledge of the internal structure of the system is not available, the transfer function description of a system gives a mathematical relation between the input and output signals of the system. Assuming zero initial conditions, the relationship between the input u and the output y of a system can be written as $y(s) = G(s)u(s)$ where s is the Laplace transform in continuous-time (for discrete-time systems, we use the z-transform and the variable z), and G(s) is the scalar transfer function of the system. G(s) is a rational function of the indeterminate s that can be written as a ratio of two polynomials $G(s) = \frac{n(s)}{d(s)}$ where n(s) is a numerator polynomial and d(s) is a denominator polynomial in the indeterminate s. In the above description of a transfer function, it is assumed that polynomials n(s) and d(s) are relatively prime, or coprime polynomials, i.e. they have no common factor, except possibly constants. The degree of denominator polynomial d s is the order of the linear system.

When the denominator polynomial is monic, i.e. with leading coefficient equal to one, the transfer function is normalized or nominal. It is always possible to normalize a transfer function by dividing both numerator and denominator polynomials by the leading coefficient of the denominator polynomial.

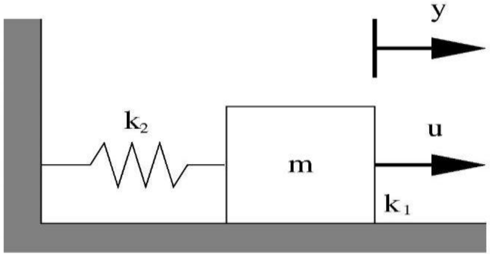

Figure 1: Mechanical system.

As an example, consider the mechanical system shown in Figure 1. For simplicity, we consider that the friction force between the floor and the mass consists of viscous friction only (we neglect the static friction and Coulomb friction). It is given by f = k\dy/dt, where ki is the viscous friction coefficient. We also assume that the displacement of the spring is small, so that the spring force is equal to k $_2$y, where k_2 is the spring constant. Applying Newton's law, the input-output description of the system from the external force u (input) to the displacement y (output) is given by $m\frac{d^2y}{dt^2} = u - k_1\frac{dy}{dt} - k_2 y$ Taking the Laplace transform and assuming zero initial conditions, we obtain $ms^2y(s) = u(s) - k_1 s y(s) - k_2 y(s)$ so that $y(s) = \frac{1}{ms^2 + k_1 s + k_2}u(s) = G(s)u(s).$ Transfer function G(s) has numerator polynomial n(s) = 1 of degree zero and denominator poly-nomial $d(s) = ms^2 + k_1 s + k_2$ of degree two. The corresponding linear system has therefore order two. Dividing both n(s) and d(s) by the leading coefficient of d(s) we obtain the normalized transfer function $G(s) = \frac{\frac{1}{m}}{s^2 + \frac{k_1}{m}s + \frac{k_2}{m}}.$

2.2. FROM TRANSFER FUNCTION TO STATE-SPACE

Similarly to network synthesis where the objective is to build a network that has a prescribed impedance or transfer function, it is very useful in control system design to determine a dynamical equation that has a described rational transfer matrix G s . Such an equation is called a realization of G s . The most common ones for linear systems are state-space realizations of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

where x {t} is the state vector, u{t} is the input, y{t} is the output and ABC are matrices of appropriate dimensions. Such realizations correspond to strictly proper transfer functions. In the case of proper transfer function, one must add a direct transmission term Du {t} to the output variable y(t). For simplicity we shall assume that D = 0 in the sequel.

For every transfer function, there are an unlimited number of state-space realizations. So it is relevant to introduce some commonly used, or canonical realizations. We shall present two of them in the sequel: the controllable form and the observable form. However, note there are other canonical forms such as the controllability, observability, parallel, cascade or Jordan form that we will not describe here for conciseness.

2.2.1. CONTROLLABLE CANONICAL FORM

For notational simplicity, we will consider a system of third order, with normalized strictly proper transfer function $G(s) = \frac{n(s)}{d(s)} = \frac{n_0 + n_1 s + n_2 s^2}{d_0 + d_1 s + d_2 s^2 + s^3}$. One can then easily extend the results to systems of arbitrary order.

The controllable canonical realization corresponding to G s has state-space matrices $A = \begin{bmatrix} -d_2 & -d_1 & -d_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = [\, n_2 \;\; n_1 \;\; n_0 \,]$.

As its name suggests, this realization is always controllable no matter whether n s and d s are coprime or not. If n s and d s are coprime, then the realization is observable as well.

2.2.2. OBSERVABLE CANONICAL FORM

The observable canonical realization corresponding to G s has state-space matrices $A = \begin{bmatrix} -d_2 & 1 & 0 \\ -d_1 & 0 & 1 \\ -d_0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} n_2 \\ n_1 \\ n_0 \end{bmatrix} \quad C = [\, 1 \;\; 0 \;\; 0 \,]$.

Note that this realization is dual to the controllable canonical realization in the sense that matrix A is transposed, and vectors B and C are interchanged. Obviously, this form is always observable. If n(s) and d s are coprime, it is also controllable.

2.3. FROM STATE-SPACE TO TRANSFER FUNCTION

Assuming zero initial conditions and taking the Laplace transform of the state-space equations we obtain that $G(s) = C(sI - A)^{-1}B$ where I denotes the identity matrix of the same dimension as matrix A. Recalling the formula of the inverse of a matrix, the above equation can be written as $C(sI - A)^{-1}B = \frac{C\mathrm{adj}\,(sI - A)B}{\det(sI - A)} = \frac{\bar{n}(s)}{\bar{d}(s)}$. Polynomial $\bar{d}(s)$ is generally referred to as the characteristic polynomial of matrix A.

It may happen that polynomials $\bar{n}(s)$ and $\bar{d}(s)$ have some common factors captured by a common polynomial term f s, so that we can write $\frac{\bar{n}(s)}{\bar{d}(s)} = \frac{n(s)f(s)}{d(s)f(s)} = \frac{n(s)}{d(s)}$ where n s and d s are coprime. The ratio of n s over d s as defined above is a representation of the transfer function G(s). When n(s) and d(s) are coprime the representation is called irreducible. It turns out that G(s) is irreducible if and only if pair (A,B) is controllable and pair {C,A) is observable.

Checking the relative primeness of two polynomials n s and d s can be viewed as a special case of finding the greatest common divisor (gcd) of two polynomials. This can be done either with the Euclidean division algorithm, or with the help of Sylvester matrices.

2.4. MINIMALITY

A state-space realization {A,B,C) of a transfer function G(s) is minimal if it has the smallest number of state variables, i.e. matrix A has the smallest dimension.

It can be proven that A B C is minimal if and only if the two polynomials defined above $\bar{n}(s) = C\mathrm{Adj}(sI - A)B$ and $\bar{d}(s) = \det(sI - A)$ ar e coprime, or equivalently, if and only if (A,B) is controllable and C A is observable.

3. CONCLUSION

We have described the use of matrix fraction descriptions (MFDs) to model scalar and multivariable linear systems. The transformation from MFDs to state-space representation motivated the introduction of several concepts and several properties specific to polynomial matrices.

There exist several extensions to the results described in this chapter. MFDs can be transformed to the so-called descriptor state-space representation $\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$ with transfer function $G(s) = C(sE - A)^{-1}B$. Matrix E may be singular, so the above representation is a generalization of the state-space form that captures impulsive dynamics and the structure at infinity. One can also mention here polynomial matrix descriptions (PMDs) $G(s) = R(s)P^{-1}(s)Q(s) + W(s)$ with the associated system polynomial matrix $\begin{bmatrix} P(s) & Q(s) \\ R(s) & W(s) \end{bmatrix}$ as a generalization of MFDs. There exists a whole theory of state-space realizations of PMDs, based on properties of the system polynomial matrix.

For practical computation with polynomials and polynomial matrices, modern software packages are available. In particular, the Polynomial Toolbox for Matlab is recommended for numerical computations with polynomials and polynomial matrices.

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