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ISOMORPHISMS AND AUTOMORPHISMS GROUPS

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Isomorphisms and Automorphisms Groups

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Abstract – We shall study the concepts of isomorphism and automorphism of group. We shall also discuss inner automorphisms. Before this topic, firstly we discuss about group, subgroup, normal subgroups. An isomorphism could also be termed as an "indirect" equality in algebraic systems. An isomorphism of a group G to itself is called automorphism. We discuss about the theorem whose state that let G be a group and let Aut (G) de not the set of all automorphism of a group G . Then Aut (G) forms a group under the composition of mapping as binary operation. After that we solve the problem and example related to this topic.

Keywords:- Groups, Subgroup, Normal Subgroups, Homomorphism, epimorphism, Monomorphism , Endomorphism

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INTRODUCTION

Definition :- A group <G , *> is a non-empty set G , together with a binary composition (operation) * on G , such that it satisfies the following postulates.

(i) Closure property for all a , b **Є** G = a * b **Є** G

(ii) Associativity : a*(b*c) = (a * b) * c , for all a , b , c **Є** G

(iii) Existence of identity : There exist an element e **Є** G such that e * a = a * e = a for all a **Є** G

(iv) Existence of inverse : for each a **Є** G , there exists an element a **Є** G such that a * a**'** = a**'** * a = e Here a**'** is called inverse of a and a**'** = a^{-1}

Since * is a binary operation on G , a * b **Є** G for all a, b **Є** G

REMARK :

1. Generally we denote the binary composition for a group by**.** (dot).

Note that a group is not just a set G. Infact a group G is made up of two entities. The set G and a binary operation**.** on G.

2. The symbol * and **.** are just notations representing binary operation.

Defination :- A group G is obtaine if its binary operation is commutative i.e. a **.** b = b **.** a for all a , b **Є** G.

Some Results Based on Groups.

In a group < G , **.** >

(i) Identity element is unique.

(ii) Inverse of each a **Є** G is unique.

(iii) a^{-1} is called inverse of a and $(a^{-1})^{-1}$ = a for all a **Є** G

(iv) $(ab)^{-1} = b^{-1} a^{-1}$, for all a , b **Є** G

(v) ab = ac => b=c (left cancellation Law)

ba = ca => b = c (Right cancellation Law)

(vi) ea = a (left identity)

ae =a (Right identity)

(vii) a^{-1} a = e (left inverse)

(viii) aa^{-1} = e (Right inverse)

(ix) A system < G , **.** > forms a group iff

(a) a (bc) = (ab) c for all a , b , c **Є** G

(b) There exist e **Є** G such that ae = a for all a **Є** G

(c) For all a **Є** G , there exist a**'** **Є** G such that aa**'** = e

Definition :- A non – empty subset H of a group G is called a subgroup of < G , **.** > if H itself is a group with respect to the some binary composition defined on G.

If G is a group with identity element e, then < G, **.** > and < {e},**.** > are called trivial subgroups of < G,**.** > Any subgroup other than these two subgroups is called a proper or non- trivial subgroup.

Definition:- A subgroup H of a group G is said to be a normal subgroup of G if Ha = aH for all a **Є** G

G and { e } are always normal subgroups of G and these group are called trivial normal subgroups . A group G = { e } not having any non-trial normal subgroups is called a simple group , e.g. H = { 1 , -1 } is a normal.

Definition:- Let < G , 0 > and < G' * > be two groups . Then a mapping f: G → G**'** is called a homomorphism, then f (G) is called homomorphism image of G.

A one to one homomorphism is called a monomorphism.

A homomorphism of a group G to itself is called an endomorphism of G.

An onto homomorphism is called an epimorphism.

Some Results Based on Homomorphisms.

(i) Let f : G→ G**'** be a homomorphism. Then

(a) f (e) = e**'**

(b) $f(a^{-1}) = (f(a))^{-1}$

(c) $f(a^n) = [f(a)]^n$, n is an integer , where e and e**'** are identity elements of G and G**'** respectively.

Definition :- A homomorphism of a group G onto a group G**'** is called an isomorphism if f is a one to one mapping.

Definition :- An isomorphism from a group G to itself is called an automorphism of G . Thus a one-one onto map f : < G , * >→ <G , * > is called an automorphism of G if f (x *y) = f (x) * f (y) for all x , y **Є** G .

Th^m :- Let G be a group and let Aut (G) denot the set of all automorphism of a group G . Then Aut (G) forms a group under the composition of mapping as binary operation.

Proof :- Let Aut (G) = {f : f is an automorphism of G}

We shall show that Aut (G) forms a group with composition of mapping as binary operation.

Closure :- If f , g **Є** Aut (G) , then gf : G → G being the composition of f and g is also one-one onto mapping.

If x ,y **Є** G then (gf) (xy) = g [f (xy)]

= g [f (x) f (y)]

= g [f(x)] [g f(y)] = (gf) (x) (gf) (y)

Thus gf is also an automorphism of G and so gf **Є** Aut (G). This shows that automorpisms holds closure.

Associativity :- Let f, g, h **Є** Aut G

(f (gh) (x) = f (gh (x)) = f (g (h (x)) --------1.

((fg)h) (x) = (fg) h (x)

= f(g(h(x)))---------2.

From eq. 1 and 2

(f(gh)(x) = ((fg)h)(x) , for all x **Є** G

f(gh) = (fg) h for all f , g, h **Є** Aut (G)

Existence of Identity: Let **I**: G →G be the identity function on G, such that I (x) = x for all x **Є** G

So I is one-one onto

Let x ,y **Є** G => xy **Є** G

= > I (xy) = xy

= I (x) I (y)

=> I (xy) = I (x) I (y)

=> I is a homomorphism.

Thus **I** : G → G is an isomorphism of G onto itself and so I **Є** Aut (G)

Existence of Inverse :- Let f **Є** Aut (G)

=> f is one-one onto mapping form G to G

= > f^{-1} is also one-one onto mapping form G to G.

Let x ,y **Є** G Then there exist x , y **Є** G , such that

$f^{-1}(x) = x_1 = f(x_1) = x$

$f^{-1}(y) = y$, $= f(y_1) = y$

Then $f^{-1}(xy) = f^{-1}[f(x_1) f(y_1)]$

$= f^{-1}(f(x , y))$ (f is homomorphism)

$= x_1 y_1 = f^{-1}(x) f^{-1}(y)$

Also f^{-1} is one-one onto

= f^{-1} is an automorphism and so $f^{-1} \in$ Aut (G)

Hence Aut (G) forms a group with respect to composite composition .

Definition:- Let G be a group and a $\in$ G . Then the mapping $T_a : G \to G$ defined by T_a (x) : a^{-1} xa is an automorphism of G and it is called an inner automorphism of additive group of integers T_a (x) = (-a) + x + (a) = x for all x $\in$ G .

Remarks:- Performing T_a on x $\in$ G is called conjugation of x by a .

2. We denote the set of all inner automorphisms of G by Inn (G)

Value Addition :- If G is an abelian group then the inner automorphism induced by a $\in$ G reduces to the identity automorphisms of G , i.e. Inn (G) = { I } . Thus, inner automorphisms are of interest mainly in case of non-abelian groups.

Example:- We now study the inner automorphism of G = D_4 the Dihedral group of symmetries of a square , induced by R_{90}

$R_{90} (R_{90}) = R_{90} R_{90} R_{90}^{-1} = R_{90}$

$R_{90}(R_{180}) = R_{90}R_{180} R_{90}^{-1} = R_{180}$

$R_{90} (R_{270}) = R_{90} R_{270} R_{90}^{-1} = R_{270}$

$R_{90}(H) = R_{90}HR_{90}^{-1}$ H

$R_{90} (V) = R_{90} V R_{90}^{-1} = V$

$R_{90} (D) = R_{90} D R_{90}^{-1}=D$

$R_{90} (D') = R_{90} D' R_{90}^{-1} = D'$

It is good exercise to see the action of inner automorphism of D_4 induced by all other elements of D_4 as well.

Problem: Let g be an element of a group G. show that the inner automorphism induced by g is same as the inner automorphism induced by Z_g, Where z is in Z (G) , the centre of G

Solution:- ϕ_{zg} (x) = (zg) x $(zg)^{-1}$ as x $\in$ G

= g(z) x ($g^{-1} z^{-1}$) , as z $\in$ Z (G)

= gz (x) $z^{-1}g^{-1}$, as z $\in$ Z (G) = $z^{-1} \in$ Z (G)

= $gxzz^{-1}g^{-1}$

= gxg^{-1}

= ϕ_g (x)

Hence $\phi_{zg} = \phi_g$

Note :- It is evident from the above problem that two distinct elements of a group G need not induce two distinct inner automorphisms , i.e. if a ≠ b in G even then a and b may be same.

Thus if G = {e, a, b, c------- } then Inn (G) = {ϕ_e , ϕ_b , ϕ_c…………} may have duplications.

Ilustration :- Aut (Z_6)

Solutions : Aut (Z_6)≈ U(6) = { 1,5 } mod 6

Thus Aut (Z_6) being a group of order Z (Prime) is cyclic and hence isomorphic to Z_2.

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