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Role of Geometric Properties of Connes' Spectral Triple

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Abstract – In this paper examples of spectral triples, which represent fractal sets, are examined and new insights into their noncommutative geometries are obtained Firstly, starting with Connes' spectral triple for a non-empty compact totally disconnected subset E of R with no isolated points, we develop a noncommutative coarse multifractal formalism. Speci µ cally, we show how multifractal properties of a measure supported on E can be expressed in terms of a spectral triple and the Dixmier trace of certain operators. If E satis µ es a given porosity condition, then we prove that the coarse multifractal box-counting dimension can be recovered. Secondly, motivated by the results of Antonescu-Ivan and Christensen, we construct a family of (1; +)-summable spectral triples for a one-sided topologically exact subshift ofnite type (µ NA ;). These spectral triples are constructed using equilibrium measures obtained from the Perron Frobenius-Ruelle operator, whose potential function is non-arithemetic and Holder continuous.

Keywords:- Noncommutative geometric, Connes, spectral triple, noncommutative integral.

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INTRODUCTION

Summary of Main Results

The main goal of this thesis is to develop the theory of noncommutative fractal geometry, as originally proposed by Connes [Con3] and Lapidus [Lap]. A summary of our main contributions towards this theory is as follows.

A Noncommutative Coarse Multifractal Formalism. We show how multifractal properties of a Borel probability measure supported on a non-empty compact fractal set E of R satisfying a certain porosity condition1 can be expressed in terms of the complementary intervals of the support of (by a fractal set we mean a non-empty totally disconnected space with no isolated points). This allows the development of a noncommutative analogue of a coarse multifractal formalism for Connes' spectral triple representation of the set E.

The Noncommutative Volume of a Subshift of Finite Type. By refining the methods of Antonescu-Ivan and Christensen given in [AIC1], we derive a (1; +)-summable spectral triple for each one-sided topologically exact subshift ofnite type (µ 1A ;) equipped with an equilibrium measure µ (where 2 C(µ 1A ;C) denotes some Holder continuous non-arithmetic potential function).

In the 1980s Connes formalised the notion of noncommutative geometry (see for instance [Con3, Con1]) and, in doing so, showed that the tools of di µ erential geometry can be extended to certain non-Hausdor µ spaces known as \bad quotients" and to spaces of a \fractal" nature. Such spaces are abundant in nature and commonly arise from various dynamical systems.

A main idea of noncommutative geometry is to analyse geometric spaces using operator algebras, particularly C µ -algebras. This idearst appeared in the work of Gelfand and Na µ µ mark [GN], where it was shown that a C µ -algebra can be seen as the noncommutative analogue of the space of complex-valued continuous functions on a locally compact metric space. Also, note that for a smooth compact spin Riemannian manifold, one can recover its smooth structure, its volume and its Riemannian metric directly from its standard Dirac operator (see [Jos]). Motivated by these observations, Connes proposed the concept of a spectral triple. A spectral triple is a triple (A;H;D) consisting of a C µ -algebra A, which acts faithfully on a separable Hilbert space H, and an essentially self-adjoint unbounded operator D de µ ned on H with compact resolvent such that the set fa 2 A : the operator [D;(a)] extends to a bounded operator de µ ned on Hg is dense in A. (Here : A ! B(H) denotes the faithful action of A on H.) Connes showed that with such a structure one can obtain a pseudo-metric on the state space S(A) of A, analogous to how the Monge-Kantorovich metric is de µ ned on the space of probability measures on a compact metric space. In 1998 Rie µ el [Rie2] and Pavlovi µ c [Pav] established conditions under which Connes' pseudo-metric is a metric and established conditions under which the metric topology of Connes' pseudo-metric is equivalent to the weak µ -topology de µ ned on S(A). Also, Connes [Con3] showed that a notion of dimension (called the metric dimension) and that a theory of integration can be derived for such structures. He also proved that for an arbitrary smooth compact spin Riemannian manifold there exists a spectral triple from which the metrical information, the measure theoretical information and the smooth structure of the manifold can be recovered (see [Con3, Ren]). This illustrates that a spectral triple allows one to move beyond the limits of classical Riemannian geometry. That is to say, not only is one able to recover classical aspects of Riemannian geometry but through the notion of a spectral triple one is able to extend the tools of Riemannian geometry to situations that present themselves at the boundary of classically de µ ned objects, for instance, objects which \live" on the boundary of Teichmuller space (such as the noncommutative torus) or those of a \fractal" nature (such as the middle third Cantor set). Although one of the original motivations for noncommutative geometry was to be able to deal with non-Hausdor µ spaces, such as foliated manifolds, which are often best represented by a noncommutative C µ -algebra (see [Con3, V µ ar, Mar, Rie3]), this new theory has scope, even when the C µ -algebra is commutative. In Connes' seminal book [Con3], the concept of a noncommutative fractal geometry is introduced. Consequently, a remarkable amount of interest has developed in this subject. The topic of [Con3], Connes gives numerous examples to indicate how fractal sets can be represented by spectral triples. Connes' examples include non-empty compact totally disconnected subsets of R with no isolated points and limit sets of Fuchsian groups of the second kind. Subsequently, in 1997 Lapidus [Lap] proposed several ways in which the notions of a noncommutative fractal geometry could be extended, after which several important articles on the subject appeared. For instance, in [GI1] Guido and Isola analysed the spectral triple presented by Connes for limit fractals in R which satisfy a certain separation condition. (Note that such sets are non-empty compact totally disconnected and have no isolated points.) There, the authors investigated aspects of Connes' pseudo-metric, the metric dimension and the noncommutative integral of Connes' spectral triple. In [GI2] this construction and analysis is extended to limit fractals in Rn, for all n 2 N. Further, Antonescu-Ivan and Christensen [AIC1] have provided a construction of a spectral triple for an AF (approximatelynite) C µ -algebra with particular focus on aspects of Connes' pseudo-metric. In [AICL] the authors give several examples of spectral triples which represent fractal sets such as the von Koch curve and the Sierpi µ nski gasket. There, the authors showed that for such sets the Hausdor µ dimension can be recovered and that Connes' pseudo-metric induces a metric equivalent to the metric induced by the ambient space on the given set. More recently, in [BP] the authors adapt Connes' spectral triple to represent the code space f0; 1gN equipped with an ultra-metric d. we begin by discussing some of the basic aspects of fractal geometry that will be required in the subsequent chapters. Therst section, Section 2.1, is split into three main parts. A general and brief introduction to fractal measures and dimensions (Subsection 2.1.1), a brief review of the Minkowski content of a subset of R (Subsection 2.1.2) andnally an introduction to the notions of coarse multifractal analysis (Subsection 2.1.3). The material contained in Subsection 2.1.1 and Subsection 2.1.2 is standard in the theory of fractal geometry and these subsections are respectively based on material contained in [Fal1] and [Fal2]. In Subsection 2.1.3, we de µ ne the coarse multifractal box-counting dimension b(q) at q 2 R for a given Borel probability measure with compact support, where we use the extension for negative q introduced by Riedi [Rie1]. We then prove that an equivalent de µ nition of b exists in terms of the complement of the support of , provided that the support of is strongly porous.

LITERATURE REVIEW

In the **1980**s Connes formalised the notion of noncommutative geometry (see for instance [Con3, Con1]) and, in doing so, showed that the tools of differential geometry can be extended to certain non-Hausdor spaces known as bad quotients" and to spaces of a \fractal" nature. Such spaces are abundant in nature and commonly arise from various dynamical systems. A main idea of noncommutative geometry is to analyse geometric spaces using operator algebras, particularly Cfi-algebras. This idea first appeared in the work of Gelfand and Nafifimark [GN], where it was shown that a Cfi-algebra can be seen as the noncommutative analogue of the space of complex-valued continuous functions on a locally compact metric space Also, note that for a smooth compact spin Riemannian manifold, one can recover its smooth structure, its volume and its Riemannian metric directly from its standard Dirac operator (see [Jos]). Motivated by these observations, Connes proposed the concept of a spectral triple. A spectral triple is a triple (A;H;D) consisting of a Cfi-algebra A, which acts faithfully on a separable Hilbert space H, and an essentially self-adjoint unbounded operator D defined on H with compact resolvent such that the set

$$\{a \in A : \text{the operator } [D, \pi(a)] \text{ extends to a bounded operator defined}$$

is dense in A. (Here fi : A ! B(H) denotes the faithful action of A on H.) Connes showed that with such a structure one can obtain a pseudo-metric on the state space S(A) of A, analogous to how the Monge-Kantorovitch metric is defined on the space of probability measures on a compact metric space. In **1998** Riefiel [Rie2] and Pavlovific [Pav] established conditions under which Connes' pseudo-metric is a metric and established conditions under which the metric topology of Connes' pseudo-metric is equivalent to the weakfi-topology defined on S(A). Also, Connes [Con3] showed that a notion of dimension (called the metric dimension) and that a theory of integration can be derived for such structures. He also proved that for an arbitrary smooth compact spin Riemannian manifold there exists a spectral triple from which the metrical information, the measure theoretical information and the smooth structure of the manifold can be recovered (see [Con3, Ren]). This illustrates that a spectral triple allows one to move beyond the limits of classical Riemannian geometry. That is to say, not only is one able to recover classical aspects of Riemannian geometry, but through the notion of a spectral triple one is able to extend the tools of Riemannian geometry to situations that present themselves at the boundary of classically defined objects, for instance, objects which live" on the boundary of Teichmuller space (such as the noncommutative torus) or those of a fractal" nature (such as the middle third Cantor set). Although one of the original motivations for noncommutative geometry was to be able to deal with non-Hausdorfi spaces, such as foliated manifolds, which are often best represented by a noncommutative Cfi-algebra (see [Con3, Vfiar, Mar, Rie3]), this new theory has scope, even when the Cfi-algebra is commutative.

Fractals, Dynamics and Renewal Theorems.

In this, we begin by discussing some of the basic aspects of fractal geometry that will be required in the subsequent chapters. The µ rst section, Section 2.1, is split into three main parts. A general and brief introduction to fractal measures and dimensions (Subsection 2.1.1), a brief review of the Minkowski content of a subset of R (Subsection 2.1.2) and µ nally an introduction to the notions of coarse multifractal analysis (Subsection 2.1.3). The material contained in Subsection 2.1.1 and Subsection 2.1.2 is standard in the theory of fractal geometry and these subsections are respectively based on material contained in [Fal1] and [Fal2]. In Subsection 2.1.3, we define the coarse multifractal box-counting dimension b(q) at q 2 R for a given Borel probability measure µ with compact support, where we use the extension for negative q introduced by Riedi [Rie1]. We then prove that an equivalent de µ nition of b exists in terms of the complement of the support of µ , provided that the support of µ is strongly porous.

Definition. (De µ nition 2.1.10.) A subset E of R is de µ ned to be strongly porous with porosity constant µ 2 (0; 1), if for each x 2 E and r 2 (0; 1] the ball B(x; r) contains a complementary interval of E with diameter greater than or equal to µ r.

Theorem. (Theorem 2.1.20.) Let µ denote a Borel probability measure on a non-empty compact subset of R. Assume that the support of µ is strongly porous with porosity constant P > 0, and let (Ik : denote the set of complementary intervals of supp(μ) whose lengths are µ nite. If µ > $k \in \mathbb{N}$2 µ □1, then for each q 2 R, we have that

$$b(q) = \inf\left\{t \in \mathbb{R} : \sum_{k \in \mathbb{N}} \mu(\overline{I}_k^{\eta})^q |I_k|^t < \infty\right\} = \inf\left\{t \in \mathbb{R} : \limsup_{N \to \infty} \frac{\sum_{k=1}^{N} \mu(\overline{I}_k^{\eta})^q |I_k|^t}{\ln(N)} = 0\right\}.$$

for each $\eta > 0, \overline{I}^{\eta}$ denote the closed ball centred at the midpoint of I with radius $(1+\eta)|I_k|/2.$ Although the result of the above theorem seems unusual at µ rst within the context of standard multifractal analysis, it is useful in the formulation of a noncommutative coarse multifractal formalism. In the next section, we introduce the concept of a one-sided subshift of µ nite type. We describe the thermodynamic formalism for this setting, as developed by Bowen and Ruelle ([Bow1, Bow2, Rue1, Rue2]). We state the results which give the existence of a Gibbs measure and the existence and uniqueness of an equilibrium measure on a one-sided topologically exact subshift of µ nite type. Finally, in Theorem, a new notion of Haar basis for the Hilbert space (Here, (µ 1A ;) denotes a one-sided topologically exact subshift of µ nite type and µ denotes a Gibbs measure with support equal to µ 1A .) This concept enables us to describe in a natural way the altration on L2(µ 1A ; B; µ) induced by the Gelfand-Na µ µ mark-Segal completion and the AF-structure of the C µ -algebra of complex-valued continuous functions de µ ned on µ 1A . Thus, we are able to re µ ne and develop the spectral triple of Antonescu-Ivan and Christensen's for an AF C µ -algebra, in the setting of a one-sided topologically exact subshift of µ nite type. The µ nal section of this chapter, iT contains a discussion of three renewal theorems for fractal sets and topologically exact subshifts of µ nite type. A description of the renewal theorems presented in [Fal3, Lal, GH] is given and it is shown how these results lead to various interesting counting results. Speci µ cally, we derive the following.

1. Let f0; 1g µ E µ [0; 1] denote a non-empty compact self-similar set whose iterated function system of similarities satis µ es the strong separation condition. Set µ equal to the Hausdor µ dimension of E and let fIk µ [0; 1] : k 2 Ng denote the set of complementary intervals of E. Let E : (0;1) ! R be de µ ned, for each r 2 (0;1), by

$$\mathcal{E}(r) := \sum_{\substack{k \in \mathbb{N} \\ |I_k| \geq r}} |I_k|^{\delta}.$$

RESULTS AND DISCUSSION

This chapter divides into two main sections: Section 4.1, a version of which has been recently published in [FS] by Falconer and Samuel and Section 4.2, which extends the results of Antonescu-Ivan and Christensen [AIC1]. In Section 4.1 we begin by describing Connes' construction of a spectral triple (A;H;D) for a non-empty compact totally disconnected subset E of R with no isolated points. Then in Subsection 4.1.1 we investigate the geometric properties of (A;H;D). Specifically, we explore the relationships between the following concepts.

1. The metric dimension of (A;H;D) and the Hausdor µ dimension dimH(E) =: µ of E

2. The noncommutative volume of (A;H;D) and the Minkowski content of E, provided that E is Minkowski measurable

3. The noncommutative volume of (A;H;D) and the measure theoretical entropy of the normalised µ -dimensional Hausdor µ measure on E with respect to S, where E is a selfsimilar set with associated iterated function system S satisfying the strong separation condition.

4. The noncommutative integral given by (A;H;D) and the normalised µ -dimensional Hausdor µ measure on E .

5. Connes' pseudo metric given by (A;H;D) and the Monge- Kantorovitch metric on the space of Borel probability measures on E (see the concluding remarks of Subsection).

BASIC CONCEPTS

In this section we set out basic terminology and notation that will frequently be encountered.

1. Let N; Z;Q;R, and C denote the sets of all natural, integer, rational, real and complex numbers, respectively. It is assumed that the natural numbers exclude zero, and so, let N0 denote the set of non-negative integers.

2. For a subset E of Rn let jEj denote the Euclidean diameter of E and let E denote the closure of E, that is, the small closed subset of Rn containing E. Further, let @E denote the closure of E minus the interior of E, where the interior of E is de µ ned to be the largest open subset of Rn which is fully contained in E.

3. For each z 2 C, the same symbol is used for the (complex) norm of z, that is, jzj := (zz)1=2. 4. Two notions which we will repeatedly use are those of comparability and asymptoticity.

4 (a) For f; g : R ! [0;1) and x0 belonging to the extended real numbers, we say that f is comparable to g as x tends to x0 if there exist constants c1; c2 > 0 such that for all x su µ ciently close to x0 (and in the case that x0 = µ 1, for all x su µ ciently large, respectively su µ ciently small), we have that c1f(x) 6 g(x) 6 c2f(x). We write f µ as x tends to x0.

(b) For f; g : R ! R and x0 belonging to the extended real numbers, we say that f is asymptotic to g as x tends to x0 (and in the case that x0 = µ 1, for all x su µ ciently large, respectively su µ ciently small) if lim x!x0 f(x)=g(x) = 1. We write f µ g as x tends to x0.

5. For a topological space (X; T) and a continuous function T : X ! X, two continuous functions g; h : X ! R are said to be cohomologous with respect to T if there exists a continuous function µ : X ! R such that g □ h = µ □ µ µ T. This difference is called the co-boundary of g and h with respect to T.

6. A topological space is called totally disconnected if and only if its connected components consist of single points. If a topological space has no open set which consists of a single point, then we say it has no isolated points.

7. Let (X; T) denote a topological space. A subset Y of X is called discrete if and only if, for all y 2 Y , there exists U 2 T such that Y \ U = fyg 8. For a topological space (X; T), let B denote the Borel µ -algebra, that is, the µ -algebra generated by the open sets of X. Two µ nite measures µ 1 and µ 2 on B are said to be equivalent, if for each B 2 B we have that µ 1(B) = 0 if and only if µ 2(B) = 0.9. Let µ denote a µ nite Borel measure on a topological space (X; T). The support of µ , denoted by supp(µ), is de µ ned to be the set of all points x 2 X for which every open neighbourhood of x has positive measure.

ANALYSIS OF THE STUDY

Here we describe and develop aspects of coarse multifractal analysis in such a way that allows for the introduction of an analogous notion within the theory of noncommutative geometry (see Subsection 4.1.2). The µ nal result of this subsection (Theorem 2.1.20) will play a vital role in the formulation of this new notion. This is a new result which allows for the calculation of the coarse multifractal box-counting dimension of the support of a measure µ on R in terms of the complement of supp(µ), provided supp(µ) is compact and strongly porous. Multifractal analysis originated from statistical mechanics and was later adapted to dynamical systems. It was developed by two independent groups of mathematicians and physicists. The fIrst approach can be traced back to the work of Mandelbrot, who in [Man1, Man2] suggested that the distribution of intermittent dissipation of energy in highly turbulent uid ows is \multifractal" in nature and studied it by calculating its moments. The second approach is due to Grassberger, Hentschel and Procaccia who in [Gra, GP, HP] generalised the work of R µ enyi [R µ en]. These two approaches were merged in the seminal paper [HMJPS].

Multifractals represent a move from the geometry of a metric space (X; d) to the geometric properties of measures supported on X. The distribution of the mass of such a measure µ may vary widely over X. By studying the local dimension of µ at each point of X, one obtains a family of sets referred to as \level sets". These are the intrinsic objects which multifractal analysis is predominantly concerned with. A number of approaches to multifractals have been developed. In what follows, we aim to introduce the coarse multifractal spectra for compact subsets of R. First we introduce the Hausdor µ dimension spectrum. Note that many of the ideas that follow can be extended to higher dimensions. However, as we are primarily interested in fractal subsets of R we state (and where necessary prove) the results for compact subsets of R.

For a µ nite Borel measure µ on R, we respectively de µ ne the lower and upper local dimension of µ at x 2 supp µ by

$$\underline{\dim}_{\mu}(x) := \liminf_{r\to 0} \frac{\ln(\mu(B(x,r)))}{\ln(r)}, \qquad \overline{\dim}_{\mu}(x) := \limsup_{r\to 0} \frac{\ln(\mu(B(x,r)))}{\ln(r)}.$$

If these coincide, we refer to the common value as the local dimension of _ at x, and denote it by dim_(x). Further, we set dim_(x) := 1 if x lies outside the support of _ and that _(x) = 0 if x is an atom of x. As a matter of interest, we note that the upper and lower local dimensions are measurable functions. This follows from the fact that they are upper and lower semi-continuous, respectively.

Since __ is a Gibbs measure and since we have that D$^{f(x) = +\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)}$(__1A) = 0, by the triangle inequality and by Parseval's identity (Theorem II:6 of [RS]), there exists a positive constant C dependant on _ such that for each k;m 2 N and a 2 A, we have that

$$\begin{aligned}
\|\pi_k(a) - \pi_{k+m}(a)\|_\infty &= \sup_{x\in\Sigma_A^\infty}\left\{\left|\sum_{l=k}^{k+m}\sum_{\omega\in\Sigma_A^l}\sum_{i=1}^{\alpha(\omega)-1}\left(\int_{\Sigma_A^\infty} a\cdot e_{\omega,i}\,d\mu_\phi\right)e_{\omega,i}(x)\right|\right\}\\
&\leqslant \sum_{l=k}^{k+m}\sum_{\omega\in\Sigma_A^l}\frac{C}{\sqrt{\mu_\phi([\omega])}}\sum_{i=1}^{\alpha(\omega)-1}\left|\int_{\Sigma_A^\infty} a\cdot e_{\omega,i}\,d\mu_\phi\right|\\
&\leqslant \sum_{l=k}^{k+m}\sum_{\omega\in\Sigma_A^l}\frac{M^{1/2}C}{\sqrt{\mu_\phi([\omega])}}\left(\sum_{i=1}^{\alpha(\omega)-1}\left|\int_{\Sigma_A^\infty} a\cdot e_{\omega,i}\,d\mu_\phi\right|^2\right)^{1/2}\\
&\leqslant \sum_{l=k}^{k+m}\sum_{\omega\in\Sigma_A^l}\frac{C\cdot M^{1/2}\cdot C_k(\alpha(\omega)-1)}{\mu_\phi([\omega])}\left(\sum_{i=1}^{\alpha(\omega)-1}\left|\int_{\Sigma_A^\infty} a\cdot e_{\omega,i}\,d\mu_\phi\right|^2\right)^{1/2}\\
&= C\cdot M^{1/2}\cdot C_k\cdot\sum_{l=k}^{k+m}\left\|P_l[D_{\mu_\phi},\pi(a)]\chi_{\Sigma_A^\infty}\right\|_{L^2}\\
&= C\cdot M^{1/2}\cdot C_k\cdot\left\|\sum_{l=k}^{k+m}P_l[D_{\mu_\phi},\pi(a)]\chi_{\Sigma_A^\infty}\right\|_{L^2}\\
&\leqslant C\cdot M^{1/2}\cdot C_k\cdot\left\|[D_{\mu_\phi},\pi(a)]\chi_{\Sigma_A^\infty}\right\|_{L^2}\\
&\leqslant C\cdot M^{1/2}\cdot C_k\cdot\|[D_{\mu_\phi},\pi(a)]\|.
\end{aligned}$$

Further, by using the fact that D__(__1A) = 0, applying the triangle inequality and applying Parseval's identity (Theorem II:6 of [RS]), for each k 2 N and a 2 C(_1A ;C)

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