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Role of Operator Algebras Arising From Sub Product Systems

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Abstract – In this paper we bring together several techniques in the theory of non-self-adjoint operator algebras and operator systems. We begin with classification of non-self-adjoint and self-adjoint operator algebras constructed from C*-correspondence and more specifically, from certain generalized Markov chains. We then transitions to the study of noncommutative boundaries in the sense of Arveson, and their use in the construction of dilations for families of operators arising from directed graphs. Finally, we discuss connections between operator systems and matrix convex sets and use dilation theory to obtain scaled inclusion results for matrix convex sets. We begin with classification of non-self-adjoint operator algebras. In Chapter 3 we solve isomorphism problems for tensor algebras arising from weighted partial dynamical systems.

Keywords:- Operator Algebras, Self-Adjoint Operator, Non-Self-Adjoint, Isometric etc.

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INTRODUCTION

The study of operator algebras, and particularly C_-algebras, has been a very active area of research in recent years, spearheaded by Elliott's classfication program for large classes of simple C_-algebras. Structural results for operator algebras often establish connections to classical dynamical theories. One very good example for this phenomenon is the work of Elliott in and on classification of approximately finite dimensional and real-rank zero circle C_-algebras in terms of K-theory. Using this work, Giordano, Putnam and Skau [56] were able to classify Cantor minimal Z-systems in terms of orbit equivalence. However, operator algebras need not be simple nor self-adjoint in general, yet they still yield many interesting invariants for the underlying dynamics. and deals with non-commutative boundaries for different classes of non-self-adjoint operator algebras along with the classification of these operator algebras and their associated boundaries. In the realm of real algebraic geometry and convex optimization, many applications were found for matricial domains defined by a linear matrix inequality, especially in the work of Helton, Klep and McCullough. A good instance of this is the solution of Helton, Klep, McCullough and Schweighofer [64] to the matrix cube problem in optimization, which was considered by Ben-Tal and Nemirovski (New Series, 2004). More precisely, using dilation, they find optimal scales fi (m) such that for any LMI domain DB, with B comprised of matrices whose ranks are at most m, and [fi1; 1]d _ DB(1), we have that every contractive d-tuple X = (X1; :::;Xd) belongs to fi (m)DB. On the other hand, in operator algebras, CP maps are the fabric for exactness, amenability and nuclearity-type properties.

In Kavruk, Paulsen, Todorov and Tomforde systematically study nuclearity related properties of operator systems, and relate them to many important problems from quantum information theory and operator algebras. One such problem is the well-known Connes' We next give an introduction for this thesis, starting with the first part. Aside from this introduction, more specific details can be found in the introductions to any of the non-preliminary chapters.

System M in a C*-cover (ι, B) is that for every ∗-representation π : B → B(H), the restriction π|M has the unique extension property. In particular, since the Shilov ideal JM is contained in the intersection of kernels of all ∗-representations, it must be trivial, so that the C*-envelope must be B = C∗(M).

BOUNDARY THEORY FOR NON-UNITAL ALGEBRAS

We explain how to define the notions of maximality and the unique extension property for representations of non-unital operator algebras, in a way that yields essentially the same theory as in the unital case. For an operator algebra A, we will say that a map ϕ : A → B (H) is a representation of A if it is a completely contractive homomorphism.

If A ⊆ B (H) is a nonunital operator algebra generating a C*algebra B=C∗(A), atheorem of Meyer [88, Section 3] (see also [18, Corollary 2.1.15]) states that every representation ϕ : A →B(K) extend s to a unital representation ϕ1 on the unitization A1 = A⨁CIHof A by specifying ϕ1(a + λIH) = ϕ(a) + λIK. T his theorem allows one to show that everyrepresenta

tion φ has a completely contractive and completely pos itive extension to B via Arveson's extension theorem. In
fact, this is a version of Arveson's extension theorem for non unital operator algebras. Meyer's theorem also shows that A has a unique (one-point) unitization, in the sense that if (ı, B) is a C*cover for the operator algebra A, and B⊆B (H) is some faithful representation of B, then the operator algebraic structure on A1 ∼ ı(A) + C1H is independent of the C*-cover and the faithful representation of B.

Next, we discuss how to extend the notions of maximality and the unique extension property to non unital operator algebras.

Definition 2.1.1. Let A ⊆ B(H) be an operator algebra g enerating a C*-algebra B.

1. We say that a representation ρ : A → B (K) has the unique extension property (UEPfor short) if every completely contractive and completely positive map π : B → B(K) extending ρ is a ∗-representation.

2. We say that a representation ρ : A → B (K) is maximal if whenever π is a represen-tation dilating ρ, then π = ρ ⊕ ψ for some repr esentation ψ.

Remark 2.1.2. When the maps in the definitions above are not assumed multiplicative, there are instances where the UEP is satisfied vacuously. We thank Rapha¨el Clou^atre for bringing these issues to our attention.

Indeed, Suppose A is a non-unital operator algebra containing a self-adjoint positive element P and let ρ : A → B be a completely contractive homomorphism. The map –ρ is completely contractive, but cannot be extended to a completely contractive completely positive map on B = C∗(A), as ρ must send P to −P . Hence, −ρ vacuously has the UEP.

Furthermore, when ρ is not maximal, the map –ρ is a completely contractive map that admits a non-trivial completely contractive dilation, coming from the one for ρ. Hence, −ρ is also not maximal. Thus, we see that if we drop the multiplicativity assumptions on our definitions above, the UEP and maximality would not be equivalent.

By a similar proof to [10, Proposition 2.2], and by the Arveson extension theorem for non-unital operator algebras via Meyer's theorem, we get that maximality is equivalent to the UEP.

Consequentially, since maximality does not depend on the choice of C*-cover, the unique extension property for representations does not depend on the choice of C*-cover, even for non-unital operator algebras. We will often refer to this fact as the" invariance of the UEP ".

For a representation ρ it is easy to see that ρ is maximal if and only if ρ1 is maximal. Hence, as maximality is equivalent to the UEP, we see that a representation ρ on A has the UEP if and only if its unitization ρ1 has the UEP. Suppose A is an operator subalgebra of B (H), and ρ: A → B (K) is a representation. We can write ρ := ρnd ⊕ 0(α), where 0 : A → C is the zero map and α is some multiplicity, such that ρnd is the non-egenerate part in the sense that ρnd(a) = ρ(a)|L with L := C∗(ρ(A))K.

When A is unital, we get that any completely contractive completely positive extension of 0 : A → C to B = C∗(A) must be 0. As the direct sum of representations with the UEP still has the UEP, we see that ρ has the UEP if and only if the unital representation ρnd has the UEP. In the case where A is separable, non-unital and contains a positive approximate identity, we let 01 : A1 → C be the unitization of the zero map, which is a unital representation.

Since this map extends uniquely to a map on the operator system S = A1 + (A1)∗, which we still denote by 01, and as A ∩ A∗ contains a positive approximate identity, by [12, Theorem 6.1] we see that 01 has the UEP when restricted to A1. Hence, the restriction 0 = 01|A has the UEP.

Hence, if we assume that A is separable and has a positive approximate identity, we still have that ρ has UEP if and only if ρnd has UEP. These assumpti ons will be satisfied by all non-unital operator algebras discussed in this paper.

The C*-envelope of a non-unital operator algebra can also be computed from the C*-envelope of its unitization. More precisely, as the pair (C∗(A), ı) where C∗(A) is the C*-

Subproduct systems and their operator algebras C*-correspondences

We assume that the reader is familiar with the basic theory of Hilbert C*-modules, which can be found in [84, 85, 97]. We only give a quick summary of basic notions and terminology as we proceed, so as to clarify our conventions.

Definition 2.2.1. Let A be a C*-algebra, E is called an inner product module over A if it is a right A-module, with an A-valued inner product ·, · on E × E, such that the following conditions are satisfied for all x, y, z ∈ E, λ ∈ C and a ∈ A.

1. A-linearity in the second variable:

x, y + λz = x, y + λ x, z , x, ya = x, y a;

2. Hermitian symmetry : x, y = y, x ∗;

3. Positivity: x, x ≥ 0;

4. Definiteness: x, x = 0 implies x = 0.

If E is an inner product module over A, then a norm on E is given by ||x|| = || x, x ||1/2, and if E is complete with respect to this norm, then E is called a Hilbert C*-module over A.

Let E and F be Hilbert C*-modules over A, and let T: E→F be a map. Then T is called adjointable if there is a map T ∗ : F → E such that for all x ∈ E and y ∈ F, y, T x = T ∗y, x. Unlike in the Hilbert space case, not all bounded linear maps on a Hilbert C*-module are adjointable. The set of all adjointable maps from E to F is denoted by L(E, F), and we denote L(E):= L(E, E) the adjointable operators on E. An adjointable map is automatically a bounded A-module map by the Uniform Boundedness Principle.

Definition 2.2.2. Let A be a C*-algebra and E a Hilbert C*-module over A. If in addition, E has a left A-module structure given by a ∗-homomorphism φ: A → L(E), we call E a C*-correspondence over A. We will say that E is faithful if φ is faithful, and that E is essential if φ (A) E = E.

Subproduct systems

The following is a C*-algebraic version of [113, Definition 1.1] for the semigroup N. It was also given in [118, Definition 1.4] for essential C*-correspondences.

Definition 2.2.7. Let A be a C*-algebra, let X = {Xn}n∈N be a family of C*-corres-pondences over A and let U = {Un,m : Xn ⊗ Xm → Xn+m} be a family of bounded bimodule maps. We will say that (X, U) is a subproduct system over A if the following conditions are met:

1. X0 = A.

2. The maps U0,n and Un,0 are given by the left and right actions of A on Xn respectively.

3. Un,m is an adjointable coisometric map for every non-zero n, m ∈ N.

4. For every n, m ∈ N we have the associativity identity Un+m,p (Un,m ⊗ IdXp) = Un,m+p (IdXn ⊗ Um,p).

In case the maps Un,m are unitaries for non-zero n, m ∈ N, we say that X is a product system. Example 2.2.8. If E is a C*-correspondence over A, define Prod (E) = {Prod (E)n} by E E E⊗(n+m) when n, m are non-zero. Then (P rod(E), U E) is a product system.

Example 2.2.9. Let H be a Hilbert space as a C*-correspondence over C. Let pn be the projection of H⊗n onto the symmetric subspace of H⊗n given by pn (ξ1 ⊗ ... ⊗ ξn) = 1n! σ∈ ξσ−1(1) ⊗ ... ⊗ ξσ−1(n).

$$V_{n+m} \circ U_{n,m} = U_{n,m} \circ (V_n \otimes V_m).$$

Definition 2.2.10. Let (X, U X) and (Y, U Y) be two subproduct systems over A and B respectively. A family V = {Vn}n∈N of maps Vn: Xn → Yn is called a morphism of subproduct systems from (X, U X) to (Y, U Y) if

1. The map ρ := V0 : A → B is a *-isomorphism,

2. For all n=0 the map Vn are uniformly bounded ρbimodule morphisms in the sense that supn∈N Vn < ∞,

3. For all n, m ∈ N the following identity hold: X Y

When the family V is a family of

1. ρ-isomorphisms, such that V −1: = {Vn−1} is a morphism from (Y, U Y) to (X, U X), we say that X and Y are isomorphic via V and write X ∼ Y .

2. ρ-unitaries, we say that X and Y are unitarily isomorphic via V and write X∼Y .

We next show that that whenever (X, U) is a product system, it is in fact unitarily isomorphic to a product system of the form (Prod (E), U E) as in Example 2.2.8, for the C*-correspondence E = X1, and that any isomorphism V = {Vn} between product systems is determined by V1.

Proposition 2.2.11. Let (X, U) be a product system over a C*-algebra A. Then (X, U) is unitarily isomorphic to (Prod(X1), U^{X1}). Furthermore, if (Prod(E), U E) and (Prod(F), U F) are product systems, and V = {Vn} an isomorphism/unitary isomorphism between them.

Then Vn = W ⊗n for a ρ-similarity/ρ-unitary W respectively, where W=V1 and ρ=V0.

Proof. We construct a morphism of subproduct systems W: (Prod(X1), U X1) → (X, U) comprised of Idunitaries {Wn : X⊗n → Xn} which, by associativity of U = {Un,m}, are uniquely determined inductively by the equations W1=IdX1 and Wn+m = Un,m ◦(Wn ⊗ Wm). Each Wn is an Idunitary, and by their inductive definition they intert wine the associativity unitary U^{X1} and U . Hence, (Prod(X1), U^{X1}) and (X, U) are unitarily isomorphic.

Stochastic matrices

We next discuss some of the preliminaries on stochastic matrices, and the results in [42] for subproduct systems associated to stochastic matrices. For the basic theory of stochastic matrices and Markov chains on discrete spaces, we recommend [48, Chapter 6] and [112]. Definition 2.3.11. Let Ω be a countable set. A stochastic matrix is a function P:Ω×Ω → R+ such that for all I ∈ Ω we have j∈Ω Pij = 1. Elements of Ω are called states of P. To every stochastic matrix, one can associate a set of edges Gr(P): = {(i, j)| Pij >0} and a {0, 1} – adjacency matrix Adj (P) representing the directed graph of P as an incidence matrix by way of Adj (P) ij =: Pij = 0 : Pij > 0 Many dynamical properties of P can be put in terms of the directed graph QP := (Ω, Gr(P), r, s) of P , where s(i, j) = I and r(i, j) = j. We note immediately that in the context of stochastic matrices in this subsection, and in Chapter 4, we take reversed range and source convention to the one taken. In and the definition of the graph of a Markov-Feller operator as in paper.

Extension theory

We recall some facts from the theory of primitive ideal spectra and extension theory for C*-algebras. More details on primitive ideal spectra of C*-algebras can be found in paper and other For an account on the Busby invariant and extension Let A be a C*-algebra. We denote by A the collection of unitary equivalenc classes of irreducible representations of A. On the other hand, we define Prim (A) to be the set of primitive ideals of A, where a primitive ideal is the kernel of an irreducible representation of A.

The set Prim (A) comes equipped with a lattice structure determined by set inclusion. Next, since any two unitarily equivalent *representations have the same kernel, the map It turns out that a C*-algebra is type I if and only if the above map κ is a injective [57].

This means that up to unitary equivalence, an irreducible representation π is completely determined by its kernel Ker π. We define SSP (H) = {SSP (H)n} by SSP (H)n = pn(H⊗n), with subproduct maps Un,m : SSP (H)n ⊗ SSP (H)m → SSP (H)n+m are given by Un,m(x ⊗ y) = pn+m(x ⊗ y). Then (SSP (H), U) is a sub product system which is not a product system.

Operator system axiomatics

We recall some definitions and results about operator system structures on Archimedean ordered unit spaces, as discussed in the work of Paulsen, Todorov and Tomforde [100]. A ∗-vector space is a complex vector space V together with a map ∗: V→V such that (v∗)∗ = v and (λv + w)∗ = λv∗ + w∗ for all v, w ∈ V and λ ∈ C. We will denote by Vsa: = {x ∈ V |x∗ = x} the Hermitian/self-adjoint elements in the ∗-vector space V.

An ordered ∗-vector space is a pair (V, V+) such that V is a ∗-vector space and V+is a cone in Vsa such that V+∩−V+= {0}. This induces a partial order on V by specifying a ≤ b if and only if b−a ∈ V+. Such a cone V+ is called the cone of positive elements in V.

For an ordered ∗-vector space (V, V+), we call an element e ∈ V an order unit if for all v ∈ Vsa there is r>0 such that re ≥ v. If additionally we have that re+v≥0 for all r>0 implies v≥0 we say that e is Archimedean.

When e ∈ V is an Archimedean order unit for an ordered ∗-vector space (V, V+), we will call the triple (V, V+, e) an Archimedean ordered *-vector space or AOU space for short.

When (V, V+, e) is an AOU space, we may define the order norm on Vsa via v = inf {t ∈ R | − te ≤ v ≤ te}.

It was shown in [101] that can be extended to a norm on V, but even though this extension is not unique, all such extensions yield equivalent norms. We call the topology induced by any extension of to a norm the order topology induced from V+ on V.

Let (V, V+, e) be an AOU space. We denote by V the collection of continuous linear functionals f: V→C with the order topology on V induced by V+. We may the define a operation f → f ∗ ∈ V given by f ∗(v) = f (v∗). This turns V into a ∗-vector space. The set V+⊆V of all positive linear functionals contains the set S(V) of states on Vcomprised of those positive linear functionals f such that f (e) = 1.

When V is a ∗-vector space, the set of all n×n matrices Mn(V) also becomes a ∗-vectors space with the operation [vij]∗ = [vji] for vij ∈ V. We say that P:= {Pn} is a matrix ordering for V if (Mn(V), Pn) is an ordered ∗-vector space, and for every n, m ∈ N and X ∈ Mn,m we have X∗PnX⊆Pm, and we call (V, P) a matrix ordered space. Given a matrix ordering {Pn} on a ∗-vector space we will say that e ∈ V is a matrix ordered unit if en=diag (e, e, ...e) is an order unit for (Mn(V), Pn) for all n ∈ N and that it is matrix Archimedean order unit if each en is an Archimedean order unit for (Mn(V), Pn) for all n ∈ N. When (V, {Pn}) and (W, {Rn}) are matrix ordered spaces, we say that a linear map φ: V→W is completely positive if for each [vij] ∈ Pn we have that [φ(vij)] ∈ Rn. We say that φ is a complete order isomorphism if φ is bijective with a completely positive inverse.

Definition 2.4.7. A triple (V, {Pn}, e) is called an (abstract) operator system if (V, {Pn}) is a matrix ordered space, and e is a matrix Archimedean order unit for it.

REFERENCES:-

Aharon BenTal and Arkadi Nemirovski (2002). On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. SIAM J. Optim., 12(3): pp. 811–833.

Bruce Blackadar (1998). Ktheory for operator algebras, volume 5 of Mathematical Sciences Research Institute Publications. Cambridge University Press, Cambridge, secondedition.

C˘alin Grigore Ambrozie and Aurelian Gheondea (2015). An interpolation problem for com pletely positive maps on matrix algebras: solvability and parametrization. Linear Multilinear Algebra, 63(4): pp. 826–851.

David P. Blecher and Christian Le Merdy. Operator algebras and their modules—an operator space approach, volume 30 of London Mathematical Society Monographs.

David P. Blecher and Vern I. Paulsen (1991). Explicit construction of universal operator algebras and applications to polynomial factorization. Proc. Amer. Math. Soc., 112 (3): pp. 839-850, 135.

David P. Blecher, Zhongin Ruan, and Allan M. Sinclair (1990). A characterization of operator algebras. J. Funct. al., 89 (1): pp. 188–201. 10, 133.

Florin Boca (1991). Free products of completely positive maps and spectral sets. J. Funct. Anal., 97 (2): pp. 251–263. 6, 123, 137.

Michael F. Barnsley (1993). Fractals everywhere. Academic Press Professional, Boston, MA, second edition, 1993. Revised with the assistance of and with a foreword by Hawley Rising, III. 51.

Michael F. Barnsley (2006). Superfractals. Cambridge University Press, Cambridge, 51.

New Series (2004). The Clarendon Press, Oxford University Press, Oxford, 2004. Oxford Science Publications. 10, 12, 14, 18.

P. Ara and K. R. Goodearl (2011). C∗-algebras of separated graphs. J. Funct. Anal., 261(9): pp. 2540–2568.

Scott Armstrong, Ken Dykema, Ruy Exel, and Hanfeng Li. On embeddigs of full amalgamated free product C∗-algebras. Proc. Amer. Math. Soc., 132 (7): pp. 2019–2030.

William B. Arveson (1976). An invitation to C∗-algebras. Springer-Verlag, New York-Heidelberg, 1976. Graduate Texts in Mathema tics.

William B. Arveson (2011). The noncommutative Choquet boundary II: hyperrigidity. IsraelJ. Math., 184: p. 34985.

William B. Arveson and Keith B. Josephson (1969). Operator algebras and measure preserving automorphisms. II. J. Functional Analysis, 4: pp. 100–134.

William B. Arveson (1969). Subalgebras of C∗-algebras. Acta Math., 123: pp. 141–224.

William B. Arveson (1972). Subalgebras of C∗-algebras. II. Acta Math., 128 (3-4): pp. 271–308.

William B. Arveson (1967). Operator algebras and measure preser ving automorphisms. Acta Math., 118: pp. 95–109.

William B. Arveson (2008). The noncommutative Choquet boundary. J. Amer. Math. Soc., 21 (4): pp. 1065–1084.

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