Isomorphism's of Tensor Algebras Arising from Weighted Partial Systems

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Abstract – Theo perator algebras associated to dynamical/topological/analytic objects, and their classification via these objects, have been the subject of study by many authors for almost 50 years, beginning with the work of Arveson [4] and Arveson and Josephson [13]. The main theme of this line of research, as is the main theme of this Paper, is to identify the extent to which the dynamical objects classify their associated non-self operator algebras. We shall mainly focus on classification of non-self-adjoint adjoint tensor operator algebras arising from a single C*-correspondence over a commutative C*-algebra, although a profusion of results have been obtained in other contexts to mention only some.

Keywords:- Tensor Algebras, C*-Algebra, Isometric Isomorphisms, conjugacy etc.

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INTRODUCTION

In this chapter, which is based on, we provide classification results for tensor algebras arising from weighted partial systems (WPS for short). Our objective is to show that WPS yield tensor algebras that are still completely classifiable up to bounded/isometric isomorphisms, while covering many examples of such classification results. For instance, those for multiplicity free finite directed graphs and for Peters' semi-crossed product.

A weighted partial system on a compact space X is a pair (σ, w) of d-tuples (σ1, ..., σd) and (w1, ..., wd) of partially defined continuous functions σi: Xi → X and wi : Xi → (0, ∞) for Xi clopen. WPS generalize many classical constructions such as non-negative matrices, continuous function on a compact space, multivariable systems, distributed function systems, graph directed systems and more.

To each WPS (σ, w) we associate a multiplicity free topological quiver (in the sense of) that encodes some information on it. This topological quiver gives rise to a C*-correspondence C(σ, w), as constructed in. We completely characterize these C*-correspondences up to unitary isomorphism and similarity, in terms of conjugacy relations between the WPS that we call branch-transition conjugacy and weighted-orbit conjugacy respectively.

We then associate a tensor algebra T+(σ, w) to C (σ, w) as one usually does for general C*-correspondences, which coincides with T+(Prod(C(σ, w))) as in subsection 2.2.3. Characterization of the C*-correspondences allows for classification of these tensor algebras up to isometric/bounded isomorphism and in some cases up to algebraic isomorphism, in terms of the WPS (σ, w). The following are our main results (See Theorems 3.5.6 and 3.5.7). Suppose (σ, w) and (τ, u) are WPS over compact spaces X and Y respectively.

1. T+(σ, w) and T+(τ, u) are isometrically isomorphic if and only if C(σ, w) and C(τ, u) are unitarily isomorphic if and only if (σ, w) and (τ, u) are branch-transition conjugate.

2. T+(σ, w) and T+(τ, u) are boundedly isomorphic if and only if C(σ, w) and C(τ, u) are similar if and only if (σ, w) and (τ, u) are weighted-path conjugate. If in addition the clopen sets Xi (which are the domains of each σi) cover X, the above is equivalent to having an algebraic isomorphism between T+(σ, w) and T+(τ, u).

The solution to these isomorphism problems require an adaptation of a new method in the analysis of character spaces due to Davidson, Ramsey and Shalit in, used in the solution of isomorphism problems of universal operator algebras associated to tuples of operators subject to homogeneous polynomial constraints.

One of the main thrusts of the work in this chapter is the use of these classification results to show that, in general, the (completely) isometric isomorphism and algebraic/(completely) bounded isomorphism problems are distinct in the sense that they require separate criteria to be solved (See Example 3.5.8).

This chapter contains six sections, including this introductory section. In Section 3.2 we introduce the notion of a weighted partial system and define three different notions of conjugacy between WPS called branch-transition conjugacy, weighted-orbit conjugacy and graph conjugacy. We then associate a C*-correspondence to every WPS in such a way that the three conjugacy relations above correspond to unitary isomorphism, similarity and isomorphism between the C*-correspondences. We give examples that show that these three conjugacy relations are distinct.

Weighted partial systems

We define the notion of weighted partial system, and examine it associate a completely positive map and a topological quiver.

Definition 3.2.1. Let X be a compact space. Advariable weighted partial system (WPS for short) is a pair (σ, w) where σ = (σ1, ..., σd) is comprised of continuous maps σi: Xi → X where each Xi is clopen in X, and w = (w1, ..., wd) is comprised of continuous non-vanishing weights wi : Xi → (0, ∞).

When wi = 1 for all 1 ≤ I ≤ d, then the information on the weights is redundant, and in this case we replace (σ, 1) by σ and call it a d-variable (clopen) partial system. Partial systems were used under the name of" quantized dynamical systems" by Kakariadis and Shalit to classify tensor algebras associated to monomial ideals in the ring of polynomials in non-commuting variables (Corollary 8.12]). Weighted partial systems provide us with concrete exa mples of Markov-Feller maps and topological quivers.

1. The operator associated to (σ, w) is a positive linear map P (σ, w) : C(X) → C(X) given by P (σ, w) (f) (x) = wi (x) f (σi(x)). :x∈Xii
2. The quiver associated to (σ, w) is the quintuple Q(σ, w) = (X, Gr(σ), r, s, P (σ, w)) where Gr(σ) is the (union) cograph of σ, i.e. the union of the cographs of σi given by Gr(σ) = ∪di=1{ (σi(x), x) | x ∈ Xi }. The range and source maps are given by r(σi(x), x) = σi(x), s(σi(x), x) = x and Radon measures

Tensor algebras

In this section we relate isomorphisms of product systems to graded isomorphisms, and to semi graded isomorphism of associated tensor algebras. Definition 3.3.1. Let E and F be C*-correspondences over A and B respectively. Anisomorphisms ϕ: T+(E) → T+(F) that satisfies ϕ (T+(E)n) = T+(F)n for all n ∈ N is called graded For a C*-correspondence E, let ΨE: E → T+(E) 1 be the isometric Banach bimodule 1) not require ρ-similarities to be adjoin table in item (2) of Definition 2.2.5.

Theorem 3.3.2. Let E and F be C*-correspondences over commutative C*-algebras A and B respectively. Then,

1. If V: E → F is a ρ-similarity for some ∗-isomorphism ρ between A and B, then there exists a graded completely bounded isomorphism AdV : T+(E) → T+(F) such that AdV |A = ρ with max {AdV cb, Ad−1cb} ≤ sup V ⊗n · sup (V−1)⊗n. n∈N n∈N.
2. If ϕ: T+(E) → T+(F) is a bounded graded isomorphism, then ρϕ:= ϕ|A : A → B is a *-somorphism and Vϕ: E→F uniquely determined by SVϕ(ξ) = ϕ(Sξ) for ξ ∈ E yields a ρϕ-similarity satisfying sup (Vϕ)⊗n ≤ ϕ and sup (V−1)⊗n ≤ ϕ−1. n∈Nn∈N

Moreover, the operations (1) and (2) are inverses of each other in the sense that ϕ = AdVϕ and V = VAdV, and in particular every bounded graded isomorphism ϕ is completely bounded with ϕ cb ≤ ϕ · ϕ−1.

Isometric isomorphisms are also automatically base detecting. Indeed, let E and F be C*-correspondences over C*-algebras A and B and let ϕ: T+(E) → T+(F) be an isometric isomorphism. Since T+(F) ⊆ T (F), we can regard ϕ as a map into the Toeplitz C*-algebra.

Thus, ϕ|A: A→T (F) is an isometric homomorphism, and is hence necessarily positive and preserves the involution from A to T (F). Thus, ϕ(A) = ϕ(A)∗ ⊆ T+(F)∗ ⊆ T (F), and we must have that ϕ (A) ⊆ T+(F) ∩ T+(F)∗ = B. Thus we have in fact that ϕ (A) ⊆ B, and the symmetric argument shows that ϕ−1(B) ⊆ A, and soϕρϕ−1 is the inverse of ρϕ, and ϕ is base-detecting.

In the forthcoming definition, we relax the assumption of gradedness of an isomorphism while maintaining base-detection. The following concept of semi-gradedness appeared in the work of Muhly and Solel in section 5 of [92, Section 5] where they resolve the isometric isomorphism problem for tensor algebras arising from aperiodic C*-correspondences, and was also used in [42] to provide classification for tensor algebras arising from stochastic matrices, in terms of the matrices.

Isomorphisms

In this section we adapt a new method in the analysis of character spaces due to Davidson, Ramsey and Shalit in, and use this to construct a bounded/isometric semi-graded isomorphism from any bounded/isometric isomorphism of our tensor algebras respectively.

We then use this to provide two theorems that separately deal with classification up to bounded isomorphism and classification up to isometric isomorphism, which turn out to yield two distinct equivalences.

We first provide a criterion for automatic continuity, that will help answer the algebraic isomorphism problem for our tensor algebras, under the assumption that the union of Xi covers X, where Xi are the clopen domains of definition for σi's. We will follow the ideas of Davidson, Katsoulis and Kribs used in. For operator algebras A and B suppose we have a surjective homomorphism ϕ: A → B.

Let S(ϕ) = {b ∊ B| there is a sequence (an) in A with an→0 and ϕ (an) → b} It is readily verified that the graph of ϕ is closed if and only if S(ϕ) = {0}, hence, by the closed graph theorem ϕ is continuous if and only if S(ϕ) = {0}. The following is an adaptation of a lemma by Sinclair, the origins of which can be traced back to.

5.1 (Sinclair). Let A and B be Banach algebras and ϕ: A→B be a surjective algebraic homomorphism. Let (bn) n∊N be any sequence in B. Then there exists N ∊ N such that for all n ≥ N, b1b2...bnS(ϕ) = b1b2...bN S(ϕ) and S(ϕ)bn...b2b1 = S(ϕ)bN ...b2b1.

SCOPE OF STUDY

The fundamental structures of direct variable based math are vector spaces. A vector space over a field F is a set V together with two parallel operations. Components of V are called vectors and components of F are called scalars. The main operation, vector expansion, takes any two vectors v and w and yields a third vector 22 v + w. The second operation, scalar duplication, takes any scalar an and any vector v and yields another vector av. The operations of expansion and increase in a vector space must fulfill the accompanying aphorisms. In the rundown underneath, let u, v and w be self-assertive vectors in V, and an and b scalars in F.

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