Journal of Advances and Scholarly Researches in Allied Education
Vol. 12, Issue No. 2, January-2017, ISSN 2230-7540

An Analysis upon Some Approximation of Fixed Points through Iterative Methods

Seema Devi[1]* Dr. Ashwani Kumar[2]

[1]Research Scholar, OPJS University, Churu, Rajasthan

[2]Associate Professor, OPJS University, Churu, Rajasthan

Abstract – In this article, we manage iterative methods for approximation of fixed points and their applications. We initially talk about fixed point theorems for a non-expansive mapping or a group of non-expansive mappings. Specifically, we express a fixed point hypothesis which addressed certifiably a problem posed during the Conference on Fixed Point Theory. We manage weak and strong union theorems of Mann's compose and Halpern's write in a Banach space. At last, utilizing these results, we consider the plausibility problem by raised blends of non-expansive withdrawals and the curved minimization problem of finding a minimizer of an arched function.

---------------------------------x---------------------------------

INTRODUCTION

Give C a chance to be a nonempty shut arched subset of a genuine Hilbert space H and let f be an appropriate curved lower semi continuous function of H into $(-\infty,\infty]$. Consider a convex minimization problem

$$\min\{f(x) : x \in C\} = \alpha.$$

The number α is called an *optimal value*, *C* is called an *admissible set* and $M = \{y \in C : f(y) = \alpha\}$ is called an *optimal set.* Next, define a function $g : H \to (-\infty,\infty]$ as follows :

$$g(x) = \begin{cases} f(x), & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then, g is a proper lower semi continuous convex function of *H* into $(-\infty,\infty]$. So, we consider the convex minimization problem $\min\{g(x) : x \in H\}$,

where g is a proper lower semi continuous convex function of *H* into $(-\infty,\infty]$. For such a g, we can define a multivalued operator ∂g on *H* by

$$\partial g(x) = \{x^* \in H : g(y) \geqq g(x) + (x^*, y - x), y \in H\}$$

for all . Such a ∂g is said to be the *sub differential* of g. Let *C* be a nonempty closed convex subset of a real Hibert space *H.* Then a mapping $T : C \to C$ is called *non-expansive* on *C* if

$$||Tx - Ty|| \leqq ||x - y|| \text{ for all } x, y \in C.$$

We denote by *F)* the set of fixed point of T. Let $A \subset H \times H$. Then, we can define a multivalued operator *B* from *H* to *H* by

$$Bx = \{y \in H : (x, y) \in A\}$$

for all $x \in H$. Inversely, if *B* is a multivalued operator from *H* to *H.* then we can define a set *A* in *H* x *H* by $A = \{(x, y) : x \in H, y \in Bx\}$. So, it is natural to regard a set in *H* x *H* in the same light with a multivalued operator from *H* to *H.* Let $A \subset H \times H$. Then, we define the domain of *A* and the range of *A* as follows:

$$D(A) = \{x \in E : Ax \neq \phi\};$$
$$R(A) = \cup\{Ax : x \in D(A)\}.$$

We also define a multivalued operator A^{-1} from *H* to *H* by $A^{-1}y = \{x \in H : y \in Ax\}$ for all $y \in H$. From this definition, we have $x \in A^{-1}0 \Leftrightarrow 0 \in Ax$. An operator $A \subset H \times H$ is *accretive* if for $(x_1, y_1), (x_2, y_2) \in A$, $(x_1 - x_2, y_1 - y_2) \geqq 0$.

If *A* is accretive, we can define, for each positive λ, the resolvent $J_\lambda : R(I + \lambda A) \to D(A)$

by $J_\lambda = (I + \lambda A)^{-1}$. We know that J_λ is a non-expansive mapping. An accretive operator $A \subset H \times H$ is called *m-accretive* if $R(I + \lambda A) = H$ for all $\lambda > 0$. If $g : H \to (-\infty, \infty]$ is a proper lower semi continuous convex function, then ∂g is a m-accretive administrator. For a m-accretive administrator A, we can think about the following beginning value problem:

$$\frac{du(t)}{dt} + Au(t) \ni 0, \quad t > 0,$$
$$u(0) = x, \qquad (**)$$

where x is an element of $\overline{D(A)}$. Then, it is well known that $(**)$ has a unique strong solution $u : [0, \infty) \to H$. Putting $S(t)x = u(t)$, we know that the family $\{S(t) : t \in [0, \infty)\}$ of mappings on $\overline{D(A)}$ satisfies the following conditions:

(i) $S(t+s)x = S(t)S(s)x$ for every $t, s \in [0, \infty)$ and $x \in \overline{D(A)}$;

(ii) $S(0)x = x$ for every $x \in \overline{D(A)}$;

(iii) for each $x \in \overline{D(A)}$, $t \mapsto S(t)x$ is continuous;

(iv) $\|S(t)x - S(t)y\| \leqq \|x - y\|$ for every $x, y \in \overline{D(A)}$ and $t \in [0, \infty)$.

Such a family $\{S(t) : t \in [0, \infty)\}$ is called a *one parameter non-expansive semi group* on $\overline{D(A)}$; see Brezis. We also know that

$$0 \in \partial g(x_0) \Leftrightarrow g(x_0) = \min\{g(x) : x \in H\}$$
$$\Leftrightarrow x_0 \in \bigcap_{t \geqq 0} F(S(t)),$$

where $F(S(t))$ is the set of fixed points of $S(t)$. Further, we have that for $\lambda > 0$,

$$0 \in \partial g(x_0) \Leftrightarrow J_\lambda x_0 = x_0.$$

Accordingly, an arched minimization problem is proportionate to a fixed point problem for a non-expansive mapping or a group of non far reaching mappings. Further, we realize that one method for illuminating $(*)$ is the proximal point calculation initially presented by Martinet (1970). The proximal point calculation depends on the thought of resolvent J_λ, i.e.,

$$J_\lambda x = \arg\min\{g(z) + \frac{1}{2\lambda}\|z - x\|^2 : z \in H\},$$

introduced by Moreau. The proximal point algorithm is an iterative procedure, which starts at a point $x_1 \in H$, and generates recursively a sequence $\{x_n\}$ of points $x_{n+1} = J_{\lambda_n} x_n$, where $\{\lambda_n\}$ is a sequence of positive numbers; see, for instance, Rockafellar (1976). On the other hand, let $\{g_1, g_2, \dots, g_n\}$ be a finite family of real valued continuous arched functions on a Hilbert space H. The problem is to discover a solution of the limited raised disparity framework, i.e., to discover such a point $x \in C$ that $C = \{x \in H : g_i(x) \leqq 0,\ i = 1, 2, \dots, n\}$.

Such a problem is known as the practicality problem. This problem is likewise connected with approximation of fixed points.

In this article, we initially examine fixed point theorems for a non-expansive mapping or a group of non-expansive mappings. Specifically, we express a fixed point hypothesis which addressed positively a problem postured amid the Conference on Fixed Point Theory and Applications held at CIRM, Marseille-Luminy, 1989. At that point we examine nonlinear ergodic theorems of Baillon's write for nonlinear semi groups of non-expansive mappings. Specifically, we state nonlinear ergodic theorems which addressed certifiably the problem postured amid the Second World Congress on Nonlinear Analysts, Athens, Greece, 1996. Next, we manage weak and strong meeting theorems of Mann's compose and Halpern's write in a Banach space. At last, utilizing these results, we consider the achievability problem by arched blends of non-expansive withdrawals and the curved minimization problem of finding a minimizer of a raised function.

PRELIMINARIES

Let *C* be a nonempty closed convex subset of a Banach space *E* and let *T* be a mapping of *C* into *C.* Then we denote by *R(T)* the range of *T.* A mapping *T* of *C* into *C* is said to be *asymptotically regular* if for every $x \in C, T^n x - T^{n+1}x$ converges to 0. Let *D* be a subset

of *C* and let *P* be a mapping of *C* into *D.* Then *P* is said to be *sunny* if $P(Px + t(x - Px)) = Px$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping *P* of *C* into *C* is said to be a *retraction* if $P^2 = P$. If a mapping *P* of *C* into *C* is a retraction, then $Pz = z$ for every $z \in R(P)$. A subset *D* of *C* is said to be a *sunny non-expansive retract* of *C* if there exists a sunny non-expansive retraction of *C* onto *D.*

Let *E* be a Banach space. Then, for every ε with $0 \leq \varepsilon \leq 2$, the *modulus* $\delta(\varepsilon)$ *of convexity* of *E* is defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}.$$

A Banach space *E* is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. *E* is also said to be *strictly convex* if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$

and A uniformly convex Banach space is strictly convex.

Let *E* be a Banach space and let E^* be its dual, that is, the space of all continuous linear functionals x^* on *E.* The value of $x^* \in E^*$ at $x \in E$ will be denoted by (x, x^*). With each $x \in E,$ we associate the set $J(x) = \{x^* \in E^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$. Using the

Hahn-Banach theorem, it is immediately clear that $J(x) \neq \phi$ for any $x \in E.$ Then the multivalued operator $J : E \to E^*$ is called the *duality mapping* of *E*. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of *E.* Then a Banach space *E* is said to be *smooth* provided $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in U.$ At the point when this is the case, the standard of E is said to be Gateaux differentiable. It is said to be Frechet differentiable if for every x in {/, this cutoff is achieved consistently for y in U. The space E is said to have a consistently Gateaux differentiable standard if for each $y \in U,$ the limit is attained uniformly for $x \in U.$ It is notable that if E is smooth, at that point the duality mapping J is single valued. It is additionally realized that if E has a Frechet differentiable standard, at that point J is standard to standard continuous. A shut curved subset C of a Banach space E is said to have typical structure if for each shut bounded arched subset K of C, which contains no less than two points, there exists a component of K which isn't a diametral point of K. Baillon and Schoneberg (1981) likewise presented the following weakening of the idea of ordinary structure: A shut curved subset C of a Banach space is said to have asymptotic typical structure if for each shut bounded raised subset K of C, which contains no less than two points and each sequence $\{x_n\}$ in *K* satisfying $x_n - x_{n+1} \to 0$ as $n \to \infty$, there is a point $x \in K$ such that $\liminf_{n \to \infty} \|x_n - x\| < \delta(K),$

where $\delta(K)$ is the measurement of K. It is notable that a shut curved subset of a consistently arched Banach space has ordinary structure and a conservative raised subset of a Banach space has typical structure. A Banach space E is said to fulfill OpiaVs condition if $x_n \rightharpoonup x$ and $x \neq y$ simply

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|,$$

where $\rightharpoonup$ denotes the weak convergence to *x.* Let *S* be a semitopological semigroup, i.e., a semigroup with Hausdorff topology such that for each $s \in S,$ the mappings $t \mapsto ts$ and $t \mapsto st$ of *S* into itself are continuous. Let *B(S)* be the Banach space of all bounded real valued functions on *S* with supremum norm and let *X* be a subspace of *B(S)* containing constants. Then, an element μ of X^* is called a *mean* on *X* if $\|\mu\| = \mu(1) = 1.$ We know that $\mu \in X^*$ is a mean on *X* if and only if $\inf\{f(s) : s \in S\} \leq \mu(f) \leq \sup\{f(s) : s \in S\}$ for every $f \in X.$ A real valued function μ on *X* is called a *sub mean* on *X* if the following properties are satisfied:

(i) $\mu(f+g) \leq \mu(f) + \mu(g)$ for every $f, g \in X;$

(ii) $\mu(\alpha f) = \alpha \mu(f)$ for every $f \in X$ and $\alpha \geq 0;$

(iii) For $f, g \in X,\ f \leq g$ implies $\mu(f) \leq \mu(g);$

(iv) $\mu(c) = c$ for every constant function c.

Obviously every mean on I is a submean. The idea of submean was first presented by Mizoguchi and Takahashi (1990). For a submean μ on *X* and $f \in X,$ sometimes we use $\mu_t(f(t))$ instead of $\mu(f)$. For each $s \in S$ and $f \in B(S)$, we define elements $\ell_s f$ and $r_s f$ of B (S) given by $(\ell_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in S.$ Let X be a subspace of *B(S)*containing constants which is invariant under $\ell_s, s \in S$ (resp. $r_s,\ r \in S$). Then a mean μ on *X* is said to be *left invariant* (resp. *right invariant)* if $\mu(f) = \mu(\ell_s f)$ (resp. $\mu(f) = \mu(r_s f)$) for all $f \in X$ and $s \in S.$ An *invariant mean* is a left and right invariant mean. A sub mean μ on *X* is said to be *left sub invariant* if $\mu(f) \leq \mu(\ell_s f)$ for all $f \in X$ and $s \in S.$ Let *S* be a semi topological semi group. Then *S* is called *left* (resp. *right) reversible* if any two closed right (resp. left) ideals of *S* have non-void intersection. If *S* is left reversible, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on *S* is defined by $a \leq b$ if and only if $\{a\} \cup \overline{Sa} \supset \{b\} \cup \overline{Sb},\ a, b \in S.$ Similarly, we can define the binary relation "$\leq$" on a right reversible semi topological semi group *S.*

FIXED POINT THEOREMS

In this area, we talk about fixed point theorems for a non-expansive mapping or a group of non-expansive mappings. The main fixed point theorem for non-expansive mappings was built up in 1965 by Browder (1965). He demonstrated that if C is a bounded shut raised subset of a Hilbert space H and T is a non-expansive mapping of C into itself, at that point T has a fixed point in C. Very quickly, both Browder and Gohde demonstrated that the same is valid if E is a consistently arched Banach space. Kirk likewise demonstrated the following theorem:

Theorem 4.1 Let E be a reflexive Banach space and let C be a nonempty bounded shut raised subset of E which has ordinary structure. Give T a chance to be a non-expansive mapping of C into itself Then F(T) is nonempty.

After kirk's theorem, numerous fixed point theorems concerning non-expansive mappings have been demonstrated in a Hilbert space or a Banach space. Specifically, Baillon and Schoneberg presented the idea of asymptotic ordinary structure and generalized Kirk's fixed point theorem as follows:

Theorem 4.2 Let E be a reflexive Banach space and let C be a nonempty bounded shut raised subset of E which has asymptotic typical structure. Give T a chance to be a non-expansive mapping of C into itself. At that point F(T) is nonempty.

Then again, DeMarr demonstrated the following fixed point theorem for a commu¬tative group of non-expansive mappings.

Theorem 4.3 Let C be a minimized curved subset of a Banach space E and let S be a commutative group of non-expansive mappings of C into itself. At that point S has a typical fixed point in C, i.e., there exists $z \in C$ *such that* $Tz = z$ *for every* $T \in \mathcal{S}$.

Browder demonstrated the following fixed point theorem without conservativeness:

Theorem 4.4 Let C be a bounded shut raised subset of a consistently arched Banach space E and let S be a commutative group of non-expansive mappings of C into itself. At that point S has a typical fixed point in C

Further, we endeavor to stretch out these theorems to a noncommutative semi group of non-expansive mappings. Give S a chance to be a semi topological semi group and given C a chance to be a nonempty subset of a Banach space E. At that point a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of *C* into itself is called a *non-expansive semi group* on *C* if it satisfies the following:

(i) $T_{st}x = T_sT_tx$ for all $s, t \in S$ and $x \in C$;

(ii) For each $x \in C$, the mapping $s \mapsto T_sx$ is continuous;

(iii) For each $s \in S$, T_s is a non-expansive mapping of *C* into itself.

For a non-expansive semi group $\mathcal{S} = \{T_s : s \in S\}$ 011 C, we denote by *F(S)* the set of common fixed points of $T_s, s \in S$. Let S be a semi topological semi group, let C(S) be the Banach space of all bounded continuous functions on S and let $RUC(S)$ be the space of all bounded right uniformly continuous functions on S, i.e., all $f \in C(S)$ such that the mapping $s \mapsto r_sf$ is continuous. Then $RUC(S)$ is a closed sub algebra of $C(S)$ containing constants and invariant under ℓ_s and r_s, $s \in S$;.

In 1969, Takahashi (1969) demonstrated the main fixed point theorem for a noncommutative semi group of non-expansive mappings which generalizes DeMarr's fixed point theorem, that is, he demonstrated that any discrete left manageable semi group has a typical fixed point. Mitchell generalized Takahashi's result by demonstrating that any discrete left reversible semi group has a typical fixed point. Lau demonstrated the following theorem:

Theorem 4.5 *Let S be a semi topological semi group and let* $A(S)$ *be the space of all* $f \in C(S)$ *such that* $\{\ell_s f : s \in S\}$ *is relatively compact in the norm topology of* $C(S)$. *Let* $\mathcal{S} = \{T_s : s \in S\}$ *be a non-expansive semi group on a compact convex subset C of a Banach space E. Then* $A(S)$ *has a left invariant mean if and only if* $\mathcal{S}$ *has a common fixed point in c*.

Lim generalized Kirk's result, Browder's result and Mitchell's result by showing the following theorem:

Theorem 4.6 *Let* S *be a left reversible semi topological semi group. Let C be a weakly compact convex subset of a Banach space E which has normal structure and let* $\mathcal{S} = \{T_s : s \in S\}$ *be a non-expansive semi group on C. Then* $\mathcal{S}$ *has a common fixed point in C .*

Takahashi and Jeong also generalized Browder's result by using the concept of sub mean.

Theorem 4.7 *Let S be a semi topological semi group. Let* $\mathcal{S} = \{T_s : s \in S\}$ *be a non-expansive semigroup on a bounded closed convex subset C of a uniformly convex Banach space E. Suppose that* $RUC(S)$ *has a left, sub invariant sub mean. Then* $\mathcal{S}$ *has a common fixed point in C.* To prove Theorem 4.7, we need the following lemma:

Lemma : *Let* $p > 1$ *and* $b > 0$ *be two fixed numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function (depending on p and b)* $g : [0, \infty) \to [0, \infty)$ *such that* $g(0) = 0$ *and*

$$\|\lambda x + (1-\lambda)y\|^p \leqq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_b$ *and* $0 \leqq \lambda \leqq 1$, *where* $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$ *and* B_b *is the closed ball with radius b and centered at the origin.*

We may comment on the relationship between "$RUC(S)$ has an invariant mean" and *"S* is left reversible". As well known, they do not imply each other in general. But if $RUC(S)$ has sufficiently many functions to separate closed sets, then "$RUC(S)$ has an invariant mean" would imply "S is left and right reversible". Recently, Lau and Takahashi generalized Lim's result and Takahashi and Jeong's result.

Theorem 4.8 *Let* S *be a semi topological semi group, let C be a nonempty weakly compact convex subset* *of a Banach space E which has normal structure and let* $\mathcal{S} = \{T_s : s \in S\}$ *be a non-expansive semi group on C. Suppose* $RUC(S)$ *has a left sub invariant sub mean.*

Then $\mathcal{S}$ *has a common fixed point in C.*

To prove Theorem 4.9, we need two lemmas.

Lemma :*A closed convex subset C of a Banach space has normal structure if and only if it does not contain a sequence* $\{x_n\}$ *such that for some* $c > 0,$

$$\|x_n - x_m\| \leqq c \text{ and } \|x_{n+1} - \overline{x_n}\| \geqq c - \frac{1}{n^2}$$

for all $n \geqq 1$ *and* $m \geqq 1,$ *where* $\overline{x_n} = \frac{1}{n}\sum_{i=1}^{n} x_i.$

Lemma : *Let X be a compact convex subset of a separated topological vector space E, let* $f_1, f_2, \ldots, f_n$ *be a finite family of lower semi continuous convex functions from X into R and let* $c \in R,$ *where R denotes the set of real numbers. Then the following conditions (i) and (ii) are equivalent:*

(i) *There exists* $x_0 \in X$ *such that* $f_i(x_0) \leqq c$ *for all* $i = 1, 2, \ldots, n;$

(ii) *For any finite non-negative real numbers* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *with* $\sum_{i=1}^{n} \alpha_i = 1,$ *there exists* $y \in X$ *such that* $\sum_{i=1}^{n} \alpha_i f_i(y) \leqq c.$

Theorem 4.8 answers certifiably a problem postured amid the Conference on Fixed Point Theory and Applications held at CIRM, Marseille-Luminy, 1989, regardless of whether Lim's result and Takahashi and Jeong's result can be completely reached out to such Banach spaces for amiable semi groups. We don't know whether "ordinary structure "in Theorem 4.8 would be supplanted by "asymptotic typical structure".

WEAK CONVERGENCE THEOREMS

The main nonlinear ergodic theorem for non-expansive mappings was set up in 1975 by Baillon in the system of a Hilbert space.

Theorem 4.9 Let C be a shut arched subset of a Hilbert space H and let T be a non-expansive mapping of C into itself. In the event that the set $F(T)$ *of fixed points of T is nonempty, then for each* $x \in C,$ *the Cesaro means*

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T).$

This theorem was stretched out to a consistently raised Banach space whose standard is Frechet differentiable by Bruck.

Theorem 4.10 Let C be a shut curved subset of a consistently raised Banach space E with a Frechet differentiable standard. In the event that $T : C \to C$ *is a non-expansive mapping with a fixed point, then the Cesaro means of* $\{T^n x\}$ *converge weakly to a fixed point of T.*

In their theorems, putting $y = Px$ for each $x \in C,$ we have that *P* is a non-expansive retraction of *C* onto *F(T)* such that $PT^n = T^n P = P$ for all $n = 1, 2, \ldots$ and $Px \in \overline{co}\{T^n x : n = 0, 1, 2, \ldots\}$ for each $x \in C,$ where $\overline{co}A$ is the closure of the convex hull of A.

We talk about nonlinear ergodic theorems for a nonlinear semigroup of non-expansive mappings in a Hilbert space or a Banach space. Before talking about them, we give a definition. Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on $RUC(S).$ Then $\{\mu_\alpha \in A\}$ is said to be *asymptotically*

invariant if for each $f \in RUC(S)$ and $s \in S,$ $\mu_\alpha(f) - \mu_\alpha(\ell_s f) \to 0$ and $\mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$

Let us give an example of asymptotically invariant nets. Let $S = \{0, 1, 2, \ldots\}$ and let *N* be the set of positive integers. Then for $f = (x_0, x_1, \ldots) \in B(S)$ and $n \in N,$ the real valued function μ_n defined by

$$\mu_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} x_k$$

is a mean. Further since for $f = (x_0, x_1, \ldots) \in B(S)$ and $m \in N$

$$\begin{aligned} |\mu_n(f) - \mu_n(r_m f)| &= \left|\frac{1}{n}\sum_{k=0}^{n-1} x_k - \frac{1}{n}\sum_{k=0}^{n-1} x_{k+m}\right| \\ &\leq \frac{1}{n} \cdot 2m\|f\| \to 0, \end{aligned}$$

as $n \to \infty$ and S is commutative, $\{\mu_n\}$ is an asymptotically invariant net of means.

If *C* is a nonempty subset of a Hilbert space *H* and $\mathcal{S} = \{T_s : s \in S\}$ is a non-expansive semi group on *C* such that $\{T_s x : s \in S\}$ is bounded for some $x \in C,$ then we know that for each $u \in C$ and $v \in H,$ the functions $f(t) = \|T_t u - v\|^2$ and $g(t) = (T_t u, v)$ are in

$RUC(S)$. Let μ be a mean on $RUC(S)$. Then since for each $x \in C$ and $y \in H$, the real valued function $t \mapsto (T_t x, y)$ is in $RUC(S)$, we can define the value $\mu_t(T_t x, y)$ of μ at this function. By linearity of μ and of the inner product, this is linear in y; moreover, since

$$|\mu_t(T_t x, y)| \leq \|\mu\| \cdot \sup_t |(T_t x, y)| \leq (\sup_t \|T_t x\|) \cdot \|y\|,$$

it is continuous in *y.* So, by the Riesz theorem, there exists an $x_0 \in H$ such that $\mu_t(T_t x, y) = (x_0, y)$ for every $y \in H$. We write such an x_0 by $T_\mu x$;.

Presently we can express a nonlinear ergodic theorem for noncommutative semi groups of non-expansive mappings in a Hilbert space.

Theorem 4.11 Let C be a nonempty subset of a Hilbert space H and *let S be a semi topological semi group such that $RUC(S)$ has an invariant mean. Let $\mathcal{S} = \{T_t : t \in S\}$ be anon-expansive semi group on C such that $\{T_t x : t \in S\}$ is bounded and $\cap_{s \in S} \overline{co}\{T_{st} x : t \in S\} \subset C$ for some $x \in C$. Then, $F(\mathcal{S}) \neq \phi$. Further, for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ of means on $RUC(S)$, the net $\{T_{\mu_\alpha} x : \alpha \in A\}$ converges weakly to an element $x_0 \in F(\mathcal{S})$.*

Utilizing Theorem 4.11, we have Theorem 4.9. By a similar method, we can demonstrate the following nonlinear ergodic theorems:

Theorem 4.12 Let C be a shut curved subset of a Hilbert space H and let T be a one-parameter non-expansive mapping of C into itself. *If $F(T)$ is nonempty, then for each $x \in C$,*

$$S_r(x) = (1-r)\sum_{k=0}^{\infty} r^k T^k x,$$ *as $r \uparrow 1$, converges weakly to an element $y \in F(T)$.*

Theorem 4.13 *Let C be a closed convex subset of a Hilbert space H and let $\mathcal{S} = \{S(t) : t \in [0, \infty)\}$ be a non-expansive semi group on C. If $F(\mathcal{S})$ is nonempty, then for each $x \in C$,*

$$S_\lambda(x) = \frac{1}{\lambda}\int_0^\lambda S(t)x\,dt,$$

as $\lambda \to \infty$, converges weakly to an element $y \in F(\mathcal{S})$.

Next, we express a nonlinear ergodic theorem for non-expansive semigroups in a Banach space. Before expressing it, we give a definition. A net $\{\mu_\alpha\}$ of continuous linear functional on $RUC(S)$ is called *strongly regular* if it satisfies the following conditions:

(i) $\sup_\alpha \|\mu_\alpha\| < +\infty$;

(ii) $\lim_\alpha \mu_\alpha(1) = 1$;

(iii) $\lim_\alpha \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in S$.

Theorem 4.14 *Let S be a commutative semi topological semi group and let E be a uniformly convex Banach space with a Frechet differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a non-expansive semi group on C such that $F(\mathcal{S})$ is nonempty. Then there exists a unique non-expansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$.*

Further, if $\{\mu_\alpha\}$ is a strongly regular net of continuous linear functional on $RUC(S)$, then for each $x \in C$, $T_{\mu_\alpha} T_t x$ converges weakly to Px uniformly in $t \in S$.

We have not known whether Theorem 4.14 would hold in the case when S is noncommutative. As of late, Lau, Shioji and Takahashi tackled the problem as follows:

Theorem 4.15 Let C be a shut raised subset of a consistently curved Banach space E, let S be a semi topological semi group which $RUC(S)$ *has an invariant mean, and let $\mathcal{S} = \{T_t : t \in S\}$ be a non-expansive semi group on C with $F(\mathcal{S}) \neq 0$. Then there exists a non-expansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for each $t \in S$* and $Px \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$.

This is a generalization of Takahashi's result for an amiable semigroup of nonexpan¬sive mappings on a Hilbert space. Facilitate they stretched out Rode's result to an agreeable semigroup of non-expansive mappings on a consistently arched Banach space whose standard is Frechet differentiable.

Theorem 4.16 Let E be a consistently raised Banach space with a Frechet differentiable standard and let S *be a semi topological semi group. Let C be a closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a non-expansive semi group on C with $F(\mathcal{S}) \neq \phi$. Suppose that $RUC(S)$ has an invariant mean. Then there exists a unique non-expansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for each $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$. Further, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means on X, then for each $x \in C$, $\{T_{\mu_\alpha} x\}$ converges weakly to Px.*

To demonstrate Theorem 4.16, they utilized Theorem 4.15 and the following lemma which has been demonstrated in Lau, Nishiura and Takahashi.

Lemma : Let E be a consistently raised Banach space with a Frechet differentiable standard and let S *be a semi topological semi group. Let C be a closed convex subset of E and let* $\mathcal{S} = \{T_t : t \in S\}$ *be a non-expansive semi group on C with* $F(\mathcal{S}) \neq \phi$. *Then, for each* $x \in C, F(\mathcal{S}) \cap \cap_{s \in S} \overline{co}\{T_{ts}x : t \in S\}$ *consists of at most one point.*

The following theorem has been demonstrated in Takahashi and Lau, Nishiura and Takahashi when E is a Hilbert space.

Theorem 4.17 Let E be a consistently raised Banach space with a Frechet differentiable standard and let S *be a semi topological semi group. Ijet C be a closed convex subset of E and let* $\mathcal{S} = \{T_t : t \in S\}$ *be a non-expansive semi group on C with* $F(\mathcal{S}) \neq \phi$. *Suppose that for each* $x \in C, F(\mathcal{S}) \cap \cap_{s \in S} \overline{co}\{T_{ts}x : t \in S\}$ *is nonempty. Then there exists a non-expansive retraction P from C onto F(S) such that* $PT_t = T_tP = P$ *for each* $t \in S$ *and* $Px \in \overline{co}\{T_tx : t \in S\}$ *for each* $x \in C$.

On the other hand, Mann introduced an iteration procedure for approximating fixed points of a mapping *T* in a Hilbert space as follows: $x_1 = x \in C$ and $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$ for $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Afterward, Reich talked about this iteration technique in a consistently raised Banach space whose standard is Frechet differentiable and gotten the following theorem:

Theorem 4.18 Let C be a shut convex subset of a consistently raised Banach space E with a Frechet differentiable standard,, let $T : C \to C$ *be a non-expansive mapping with a fixed point and let* $\{c_n\}$ *be a real sequence such that* $0 \leq c_n \leq 1$ *and* $\Sigma_{n=1}^{\infty} c_n(1 - c_n) = \infty$. *If* $x_1 \in C$ *and* $x_{n+1} = c_n Tx_n + (1 - c_n)x_n$ *for* $n \geq 1$, *then* $\{x_n\}$ *converges weakly to a fixed point of T.*

This theorem has been known for those consistently arched Banach spaces that fulfill Opial's condition. Tan and Xu demonstrated the following intriguing result which generalizes the result of Reich.

Theorem 4.19 Let C be a shut arched subset of a consistently curved Banach space E which fulfills Opial's condition or whose standard is Frechet differentiable and let $T : C \to$ *C be a non-expansive mapping with a fixed point. Then for any initial data* x_1 *in C, the iterates* $\{x_n\}$ *defined by*

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n \quad \text{for } n \geq 1,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are chosen so that* $\Sigma_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, $\Sigma_{n=1}^{\infty} \beta_n(1 - \alpha_n) < \infty$, $\limsup_{n \to \infty} \beta_n < 1$, *join weakly to a fixed point of T.*

To demonstrate Theorem 4.19, Tan and Xu utilized the following two lemmas.

Lemma 4.20 Let C be a nonempty shut raised subset of a consistently arched Banach space E with a Frechet differentiable standard and let $\{T_1, T_2, T_3, \dots\}$ *be a sequence of non-expansive mappings of C into C such that* $\cap_{n=1}^{\infty} F(T_n)$ *is nonempty. Let* $x \in C$ *and put* $S_n = T_n T_{n-1} \dots T_1$ *for* $n \geq 1$. *Then, the set* $U \cap \cap_{n=1}^{\infty} \overline{co}\{S_m x : m \geq n\}$ *consists of at most one point, where* $U = \cap_{n=1}^{\infty} F(T_n)$.

Lemma *Let E be a uniformly convex Banach space, let* $\{t_n\}$ *be a real sequence such that* $0 < b \leq t_n \leq c < 1$ *for* $n \geq 1$ *and let* $a \geq 0$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences of E such that* $\limsup_{n \to \infty} \|x_n\| \leq a$, $\limsup_{n \to \infty} \|y_n\| \leq a$ *and* $\lim_{n \to \infty} \|t_n x_n + (1 - t_n)y_n\| = 0$. *Then* $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Takahashi and Kim additionally demonstrated the following theorem:

Theorem 4.21 Let E be a consistently arched Banach space E which fulfills OpiaVs condition or whose standard is Frechet differentiable, given C a chance to be a nonempty shut raised subset of E, and let $T : C \to C$ *be a non-expansive mapping with a fixed point. Suppose* $x_1 \in C$, *and* $\{x_n\}$ *is given by* $x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n$ *for all* $n \geq 1$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are chosen so that* $\alpha_n \in [a, b]$ *and* $\beta_n \in [0, b]$ *or* $\alpha_n \in [a, 1]$ *and* $\beta_n \in [a, b]$ *for some a, b with* $0 < a \leq b < 1$. *Then* $\{x_n\}$ *converges weakly to a fixed point of T.*

Roused by Theorems 4.19 and 4.21, Suzuki and Takahashi got the following theorem: Theorem 4.22 Let C be a nonempty shut raised subset of a consistently curved Banach space E which fulfills OpiaVs condition or whose standard is Frechet differentiable. Give T a chance to be a non-expansive mapping from C into itself with a fixed point. Assume that $\{x_n\}$ *is given by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n \quad \text{for all } n \geq 1,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$ *with* $\Sigma_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ *and*

$\limsup_{n\to\infty}\beta_n < 1,$ *or* $\Sigma_{n=1}^{\infty}\alpha_n\beta_n = \infty$ *and* $\limsup_{n\to\infty}\beta_n < 1.$ Then $\{x_n\}$ converges weakly to a fixed point of T.

To prove Theorem 4.22, Suzuki and Takahashi used the following two lemmas. Let *I* be an infinite subset of positive integers *N*. If $\{\lambda_n\}$ is a sequence of nonnegative numbers, then we denote by $\{\lambda_i : i \in I\}$ the subsequence of $\{\lambda_n\}$.

Lemma *Let* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *be sequences of nonnegative numbers such that* $\Sigma_{n=1}^{\infty}\lambda_n = \infty$ *and* $\Sigma_{n=1}^{\infty}\lambda_n\mu_n < \infty$. *Then for* $\varepsilon > 0$, *there exists an infinite subset I of N*

such that $\Sigma\{\lambda_j : j \in N\backslash I\} \le \varepsilon$ *and the subsequence* $\{\mu_i : i \in I\}$ *of* $\{\mu_n\}$ *converges to* 0.

Lemma 4.23 *Let* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *be sequences of non-negative numbers such that* $\lambda_{n+1} \le \lambda_n + \mu_n$ *for all* $n \in N$. *Suppose there exists a subsequence* $\{\mu_i : i \in I\}$ *of* $\{\mu_n\}$ *such that* $\mu_i \to 0, \lambda_i \to \alpha$ *and* $\Sigma\{\mu_j : j \in N\backslash I\} < \infty$. *Then* $\lambda_n \to \alpha$.

Compare Theorem 4.22 with Theorem 4.19 of Tan and Xu. This indicates that the assumption $\Sigma_{n=1}^{\infty}\beta_n(1-\alpha_n) < \infty$ in Theorem 4.12 is superfluous. We do not know whether the assumptions $\Sigma_{n=1}^{\infty}\alpha_n\beta_n = \infty$ and $\limsup_{n\to\infty}\beta_n < 1$ in Theorem 4.16 are replaced by $\Sigma_{n=1}^{\infty}\beta_n(1-\beta_n) = \infty$ and $\liminf_{n\to\infty}\alpha_n > 0.$ We also know the following strong convergence theorem which is connected with Rhoades, Tan and Xu, and Takahashi and Kim.

Theorem 4.24 *Let E be a strictly convex Banach space, let C be a nonempty closed convex subset of E, and let* $T : C \to C$ *be a non-expansive mapping which T{C) is contained in a compact subset of C. Suppose* $x_1 \in C$, *and* $\{x_n\} \subset C$ *is given by*

$x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ *for* $n \ge 1$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are chosen so that* $\Sigma_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$ *and* $\limsup_{n\to\infty}\beta_n < 1$, *or* $\Sigma_{n=1}^{\infty}\beta_n(1-\beta_n) = \infty$ *and* $\liminf_{n\to\infty}\alpha_n > 0$. *Then* $\{x_n\}$ *converges strongly to a fixed point ofT.*

Let *C* be a closed convex subset of a Banach space *E*, and let T,S be selfmaps on *C*. Then Das and Debata considered the following iteration scheme: $x_1 \in C$, and

$x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for $n \ge 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. They demonstrated a strong converence theorem concerning Roades' result. Takahashi and Tamura got the following weak convergence theorem.

Theorem 4.25 Let E be a consistently arched Banach space E which fulfills Opial's condition or whose standard is Frechet differentiable, given C a chance to be a nonempty shut curved subset of E, and let $_{\#}S.T : C \to C$ *be non-expansive mappings such that* $F(S) \cap F(T)$ *is nonempty.*

Suppose $x_1 \in C$, *and* $\{x_n\}$ *is given by* $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ *for* $n \ge 1$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are chosen so that* $\alpha_n, \beta_n \in [a,b]$ *for some* a, *b with* $0 < a \le b < 1$.

Then $\{x_n\}$ *converges weakly to a common fixed point of* S *and T.*

Further, Takahashi and Tamura obtained the following theorem:

Theorem 4.26 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and let* $S, T : C \to C$ *be non-expansive mappings such that* $F(S) \cap F(T)$ *is nonempty. Let P be the metric projection of E onto.* $F(S) \cap F(T)$ *and suppose* $x_1 \in C$, *and* $\{x_n\}$ *is given by* $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ *for* $n \ge 1$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[0,1]$. *Then* $\{Px_n\}$ *converges strongly to a common fixed point of S and T.*

To apply convergence theorems of Mann's compose to the attainability problem, we have to stretch out Theorem 4.25 to a group of limited mappings. Give C a chance to be a nonempty curved subset of a Banach space E.

Let $T_1, T_2, \dots, T_r$ be finite mappings of *C* into itself and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \leqq \alpha_i \leqq 1$ for every $i = 1, 2, \dots, r$. Then, we define a mapping *W* of *C* into itself as follows:

$$\begin{aligned} U_1 &= \alpha_1 T_1 + (1-\alpha_1)I, \\ U_2 &= \alpha_2 T_2 U_1 + (1-\alpha_2)I, \\ &\vdots \\ U_{r-1} &= \alpha_{r-1} T_{r-1} U_{r-2} + (1-\alpha_{r-1})I, \\ W = U_r &= \alpha_r T_r U_{r-1} + (1-\alpha_r)I. \end{aligned}$$

Such a *W* is called the *W-mapping* generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$.

Theorem 4.27 *Let E be a uniformly convex Banach space E which satisfies OpiaVs condition or whose norm is Frechet differentiable, let C be a nonempty closed convex subset of E, and let* $\{T_1, T_2, \dots, T_r\}$ *be finite non-expansive mappings of C into itself such that* $\cap_{n=1}^{r} F(T_n)$ *is nonempty. Let a, h be real numbers with* $0 < a \le b < 1$ *and suppose* $x_1 \in C$, *and* $\{x_n\}$ *is given by*

$$x_{n+1} = W_n x_n \quad for\ n \ge 1,$$

where W_n are W-mappings generated by $T_1, T_2, \cdots, T_r$ and $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nr} \in [a,b]$. Then $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \cdots, T_r$.

We will at last demonstrate a weak convergence theorem of Mann's compose for a non-expansive semigroup in a Banach space.

Theorem 4.28 Let E be a consistently curved Banach space E with a Frechet differentiable standard. Give C a chance to be a nonempty shut curved subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ *be a non-expansive semigroup on C such that* $F(\mathcal{S}) \neq \phi$. *Let* $\{\mu_n\}$ *be a sequence of means on* $RCU(S)$ *such that* $\|\mu_n - \ell_s^* \mu_n\| = 0$ *for every* $s \in S$. *Suppose,* $x_1 = x \in C$ *and* $\{x_n\}$ *is given By* $x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n$ *for every* $n \geq 1$,

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *If* $\{\alpha_n\}$ *is chosen so that* $\alpha_n \in [0,a]$ *for some a with* $0 < a < 1$, *then* $\{x_n\}$ *converges weakly to an element* $x_0 \in F(\mathcal{S})$.

Utilizing Theorem 4.28, we can demonstrate a weak convergence theorem of Mann's compose for a one-parameter non-expansive semigroup.

Theorem 4.29 Let E be a consistently arched Banach space E with a Frechet differentiable standard and let C be a shut curved subset of E. *Let* $\mathcal{S} = \{S(t) : t \in [0,\infty)\}$ *be a one- parameter non-expansive semigroup on C such that* $F(\mathcal{S}) \neq \phi$. *Suppose* $x_1 = x \in C$ *and* $\{x_n\}$ *is given by* $x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt$ *for every* $n \geqq 1$, *Where* $s_n \to \infty$ *as* $n \to \infty$ *and* $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *If* $\{\alpha_n\}$ *is chosen so that* $\alpha_n \in [0,a]$ *for some a with* $0 < a < 1$, *then* $\{x_n\}$ *converges weakly to a common fixed point* $z \in F(\mathcal{S})$.

STRONG CONVERGENCE THEOREMS

In this section, we examine strong convergence theorems for non-expansive mappings. Give C a chance to be a nonempty shut curved subset of a genuine Hilbert space H. In 1967, Browder got the following strong convergence theorem: For a given $u \in C$ and each $n \in N$, define a contraction $T_n : C \to C$ by

$T_n x = \left(1 - \frac{1}{n}\right) Tx + \frac{1}{n}u$ for all $x \in C$, where T is a non-expansive mapping of *C* into itself. Then, there exists a unique fixed point x_n of T_n in *C* such that

$$x_n = \left(1 - \frac{1}{n}\right) T x_n + \frac{1}{n} u.$$

Further if the set $F(T)$ of fixed points of *T* is nonempty, then $\{x_n\}$ converges strongly as $n \to \infty$ to a fixed point of T. After Browder's result, such a problem has been researched by a few creators. Specifically, Reich and Takahashi and Ueda additionally stretched out Browder's result to strong convergence theorems for resolvents of accretive administrators in a Banach space. Before expressing them, we give two definitions. A shut raised subset C of a Banach space E is said to have the fixed point property for non-expansive mappings if each non-expansive mapping of C into itself has a fixed point in each nonempty bounded shut arched subset of C with the end goal that T leaves invariant. Let A bean accretive administrator in a Banach space E. At that point An is said to fulfill the range condition if $\overline{D(A)} \subset R(I + rA)$ for every $r > 0$.

Presently we can demonstrate the principal strong convergence theorem for resolvents of accretive administrators.

Theorem 4.30 Let E be a reflexive Banach space with a consistently Gateaux differ¬entiable standard and let $A \subset E \times E$ *be an accretive operator that satisfies the range condition*.

Let C be a closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ *and every weakly compact convex subset of C has the fixed point property for non-expansive mappings. If* $0 \in R(A)$, *then for each x in C,* $\lim_{t\to\infty} J_t x$ *exists and belongs to* $A^{-1}0$.

As direct consequences of Theorem 4.30, we obtain the following two results.

Theorem 4.31 *Let E be a uniformly convex and uniformly smooth Banach space, and let* $A \subset E \times E$ *be m-accretive. If* $0 \in R(A)$, *then for each* $x \in E, \lim_{t\to\infty} J_t x$ *exists and belongs to* $A^{-1}0$.

Theorem 4.32 *Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm, let* $A \subset E \times E$ *be an accretive operator that satisfies the range condition*.

Suppose that every weakly compact convex subset of E has the fixed point ^property for non-expansive mappings. If $A^{-1}0 \neq \phi$ *and* $\overline{D(A)}$ *is convex, then for each* $x \in \overline{D(A)}$, $\lim_{t\to\infty} J_t x$ *exists and belongs to* $A^{-1}0$.

We additionally know the following theorem:

Theorem 4.33 Let C be a shut arched subset of a Banach space E and let T be a non-expansive mapping of C into itself. At that point the following hold:

(i) If $A = I - T$, then A is accretive;

(ii) $C = D(A) \subset \bigcap_{r>0} R(I + rA)$.

Theorem 4.32 generalizes Browder's strong convergence theorem. In fact, from

$$x_n = \left(1 - \frac{1}{n}\right) T x_n + \frac{1}{n} u,$$

we have $x_n + (n-1)(I-T)x_n = u$. (* * *) Putting $A = I - T$, we have from Thorem 4.33 that *A* is accretive and satisfies the range condition. Since $J_{n-1}u = x_n$ from $(***)$, we have, by Theorem 4.32, $\lim_{n\to\infty} J_n u = \lim_{n\to\infty} x_{n+1} \in (I-T)^{-1}0 = F(T)$. Recently, Wittmann dealt with the following iterative process in a Hilbert space: $x_1 = x \in C$ and. $x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n$ for $n \geq 1$, where $\{\alpha_n\}$ is a sequence. The following theorem was proved by Wittmann.

Theorem 4.34 *Let H be a Hilbert space. Let C be a nonempty closed convex subset of H. Let T be a non-expansive mapping of C into itself such that* $F(T) \neq \emptyset$. *Let* $\{\beta_n\}$ *be a sequence of real numbers such that* $0 \leq \beta_n \leq 1$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$. *Suppose that* $\{x_n\}$ *is given by* $x_1 = x \in C$ *and* $x_{n+1} = \beta_n x + (1 - \beta_n) T x_n$ *for* $n \geq 1$.

Then, $\{x_n\}$ *converges strongly to* $Px \in F(T)$, *where P is the metric projection from C onto F(T).*

Shioji and Takahashi stretched out Wittmann's theorem to a Banach space by utilizing Theo¬rem 4.30 as follows:

Theorem 4.35 Let E be a consistently raised Banach space with a consistently Gateaux differentiable standard. Give C a chance to be a nonempty shut curved subset of E. Give T a chance to be a non-expansive mapping of C into itself with the end goal that $F(T) \neq \emptyset$. *Let* $\{\beta_n\}$ *be a sequence of real numbers such that* $0 \leq \beta_n \leq 1$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$. *Suppose that* $\{x_n\}$ *is given by* $x_1 = x \in C$ *and* $x_{n+1} = \beta_n x + (1 - \beta_n) T x_n$ *for* $n \geq 1$.

Then, $\{x_n\}$ *converges strongly to* $Px \in F(T)$, *where P is aunique sunny non-expansive withdrawal from C onto F[T).*

Kamimura and Takahashi additionally got the following result by utilizing Theorem 4.30, which is associated with the proximal point calculation.

Theorem 4.36 Let E be a consistently curved Banach space with a consistently Gateaux differentiable standard and let $A \subset E \times E$ *be an m-accretive operator. Let* $x \in E$ *and let* $\{x_n\}$ *be a sequence defined by* $x_1 = x$ *and* $x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n$ *for* $n \geqq 1$, *where* $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} r_n = \infty$. *If* $A^{-1}0 \neq \phi$, *then* $\{x_n\}$ *converges strongly to an element* $Px \in A^{-1}0$, *where P -is a unique sunny non-expansive retraction of E onto* $A^{-1}0$.

Atsushiba and Takahashi demonstrated a strong convergence theorem for limited non-expansive mappings which is associated with the achievability problem.

Theorem 4.37 Let E be a consistently raised Banach space with a consistently Gateaux differ¬entiable standard. Give C a chance to be a nonempty shut arched subset of E, *let* $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nr}$ *be real numbers such that* $0 < \alpha_{ni} < 1$ *forevery* $i = 1, 2, \ldots, r-1$ *and* $n = 1, 2, \ldots, 0 < \alpha_{nr} \leqq 1$ *for every* $n = 1, 2, \cdots$ *and let* $T_1, T_2, \cdots, T_r$ *be finite non-expansive mappings of C into itself such that* $\cap_{i=1}^{r} F(T_i) \neq \emptyset$. *Let* $W_n (n = 1, 2, \cdots)$ *be the W-mappings of C into itself generated by* $T_1, T_2, \cdots, T_r$ *and* $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nr}$. *Let* $\{\beta_n\}$ *be a sequence of real numbers such that* $0 \leqq \beta_n \leqq 1$ *for every* $n = 1, 2, \cdots, \lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$. *Suppose that* $\sum_{n=1}^{\infty} |\alpha_{n+1i} - \alpha_{ni}| < \infty$ *for every* $i = 1, 2, \cdots, r$ *and* $\{x_n\}$ *is given by* $x_1 = x \in C$ *and where* $\{\beta_n\}$ *is a sequence in* $[0,1]$. *If* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *art chosen so that* $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ *and* $\lambda_n \to \infty$, *then* $\{y_n\}$ *converges strongly to the element of* $F(\mathcal{S})$ *which is nearest to* x *in* $F(\mathcal{S})$.

REFERENCES

Books –

Aamri, M., and D. El Moutawakil (2004). —Common Fixed Point Theorems for E-contractive or E-expansive Maps in Uniform Spaces.‖ *Acta Math. Acad. Paedagog* 20, no. 1: pp. 83–91.

Aksoy, G. A and Maurizi, B. (2008). Metric trees, hyperconvex hulls, and extensions, Turkish Math. J.32: pp. 219-234.

Branzei, R., Morgan, J., Scalzo V. and Tijs, S. (2003). "*Approximate fixed point theorems in Banach spaces with application in game theory*", J. Math. Anal. Appl., vol. 285, pp. 619-628.

Granas, A. and Dugundji, J. (2003). "*Fixed point theory*", Springer Monogr. Math., Springer-Verlag, New York.

J. B. Baillon and R. Sckoneberg (1981). Asymptotic normal structure and fixed points of non-expansive mappings. Proc. Amer. Math. Soc., 81, pp. 257-264.

Petric, M. A. (2010). Some results concerning cyclical contractive mappings. Gen. Math. (18)4: pp. 213–226.

Sintunavarat, W. and Kumam, P. (2012). Common fixed point theorem for cyclic generalized multi-valued mappings. *Appl. Math. Lett.* **25: pp.** 1849–1855.

Research Papers –

Aliouche, A., and C. Simpson (2011). ―Fixed Points and Lines in 2 metric Spaces.‖ *arXiv:1003.5744v3 [math.MG]*: pp. 1–26.

Chang, T. H., Chen, C. M and Huang, H. C. (2008). Generalized KKM theorems on hyperconvex metric spaces with applications, Taiwanese Journal of mathematics,(12)9: pp. 2363-2372.

Czerwik, S., Krzysztof D. and Singh, S. L. (2002). "*Round-off stability of iteration procedures for set -valued operators in b- metric spaces*", J. Natur. Phys. Sci., vol. 15, pp.1-8.

Fang, J. X. (1992), "*A note on fixed point theorem of Hadzic*", Fuzzy Sets and Systems, vol. 48, pp. 391-395.

Kirk, W.A., Srinivasan, P.S. and Veeramani, P. (2003). Fixed points for mappings satisfying cyclical contractive conditions. *Fixed Point Theory.* **(4)1:** pp. 79–89.

S. Atsushiba and W. Takahashi (1997). Approximating common fixed points of non-expansive semigroups by the Mann iteration process. *Ann. Univ. Mariae Curie-Sklodowska,* **51,** *pp. 1-16.*

T. Suzuki (2009), "A New Type of Fixed Point Theorem in Met- ric Spaces," Nonlinear Analysis, Vol. 71, 2009. pp. 5313- 5317. doi:10.1016/i.na.

Corresponding Author

Seema Devi*

Research Scholar, OPJS University, Churu, Rajasthan

E-Mail – arora.kips@gmail.com