Functional Analysis of Time Domain Boundary Element Methods: A Review

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Abstract – The present research contemplate mostly includes a survey of differing time-domain boundary element methods that can be utilized to numerically fathom the retarded potential integral equations. The point is to address the late-time stability, exactness, and computational unpredictability concerns in time-domain surface integral equation approaches. The investigation by and large focuses on the transient electromagnetic scattering of three-dimensional flawless electrically conducting bodies. Effective algorithms are produced to numerically fathom the timedomain electric, derivative electric, magnetic, and combined field integral equation for the obscure incited surface current.

Utilizing a Galerkin approach, the obscure thickness is supplanted by a piecewise polynomial approximation, the coefficients of which can be found by fathoming a straight system. The passages of the system grid of this straight system include, for the instance of a two dimensional scattering problem, integrals more than four dimensional space-time manifolds. A precise computation of these integrals is vital for the stability of this method. Utilizing piecewise polynomials of low request, the two fleeting integrals can be assessed logically, prompting part functions for the spatial integrals with entangled domains of piecewise bolster. These spatial piece functions are summed up into a class of allowable part functions. A quadrature scheme for the approximation of the two dimensional spatial integrals with acceptable part functions is displayed and demonstrated to converge exponentially by utilizing the hypothesis of countably normed spaces. From the earlier mistake gauges for the Galerkin approximation scheme are reviewed, upgraded and talked about. Specifically, the scattered wave's vitality is examined as an elective blunder measure.

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THE FOURIER AND LAPLACE TRANSFORMS

In this section, we recall the definitions of the Fourier transform and of the Laplace transform (in the distributional sense). The spaces $\mathcal{S}(\mathbb{R}^N)$ and **LT** are already used in the definitions of the respective transforms, although they are only introduced in Definition below.

For $u(t,\cdot) \in \mathcal{S}(\mathbb{R})$ and $\eta \in \mathbb{R}$, the (one dimensional) Fourier transfoim with respect, to the time variables is given by

$$\mathcal{F}_t[u(t,\cdot)](\eta,\cdot) := \int_{\mathbb{R}} e^{i\eta t} u(t,\cdot)\, dt. \qquad (1)$$

Analogously, for $u(\cdot,x) \in \mathcal{S}(\mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$, the (multi-dimensional) Fourier tmnsform with respect to the space variables is given by

$$\mathcal{F}_x[u(\cdot,x)](\cdot,\xi) := \int_{\mathbb{R}^N} e^{i\,\xi\cdot x} u(\cdot,x)\, dx. \qquad (2)$$

We sometimes write u instead of $\mathcal{F}_x[u]$. Partial Fourier transforms with respect to k spatial variables $x' := x'_k := (x_{i_1},\dots,x_{i_k})$ with $1 \le i_j < i_{j+1} \le N$ for $1 \le j \le k-1$ are defined analogously in an obvious way. In the distributional sense.

Finally, for $\omega = \eta + i\sigma \in \mathbb{C}$ and for $u(t,\cdot) \in LT$, the Fourier-Laplace tmnsform with respect to the time variables is given by

$$\mathcal{L}_t[u(t,\cdot)](\omega,\cdot) := \int_{\mathbb{R}} e^{i\omega t} u(t,\cdot)\, dt. \qquad (3)$$

We sometimes write $\hat{u}$ instead of $\mathcal{L}_t[u]$.

Remark 1 (On the Laplace and Fourier Transforms)

a) The Laplace transform, is often defined as, for $s \in \mathbb{C}$, $\mathcal{L}_t[u(t,\cdot)](s,\cdot) := \int_{\mathbb{R}} e^{-st} u(t,\cdot)\, dt.$

This definition coincides with (3) for $-s = i\omega$ or, respectively, for $\omega = is$.

b) We note that, for $\omega = \eta + i\sigma \in \mathbb{C}$,

$$\mathcal{L}_t[u(t,\cdot)](\omega,\cdot) = \mathcal{F}_t[e^{-\sigma t}u(t,\cdot)](\eta,\cdot).$$

c) For $k \in \mathbb{N}$, there holds

$$\mathcal{L}_t\left[\frac{\partial^k}{\partial t^k}u(t,\cdot)\right](\omega,\cdot) = (-i\omega)^k \mathcal{L}_t[u(t,\cdot)](\omega,\cdot).$$

We use the following obvious notation for combined space-time transforms, for example for $(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^N$,

$$\mathcal{F}_{x,t}[u(t,x)](\tau,\xi) := \mathcal{F}_{t,x}[u(t,x)](\tau,\xi) := \mathcal{F}_t[\ \mathcal{F}_x[u(t,x)](t,\xi)\ ](\tau,\xi)$$

or for $(\omega,\xi) \in \mathbb{C}\times\mathbb{R}^N$ with $\omega = \eta + i\sigma$,

$$\mathcal{F}\mathcal{L}_{x,t}[u(t,x)](\omega,\xi) := \mathcal{F}_x[\ \mathcal{L}_t[u(t,x)](\omega,x)\ ](\omega,\xi) = \mathcal{F}_{x,t}[e^{-\sigma t}u(t,x)](\eta,\xi)$$

We now return to some formal definitions. Above we have used the spaces $\mathcal{S}(\mathbb{R}^N)$ and LT without defining them. This is done in Definition 1.

Definition 1 (Tempered and Laplace Transformable Functions and Distributions)

a) Let $\|\varphi\|_{k,l} := \sup_{x\in\mathbb{R}^N}(|x|^k+1)\sum_{|\alpha|\le l}|D^\alpha\varphi(x)|$ for $k,l \ge 0$ and any multi-index α. Then $\mathcal{S}(\mathbb{R}^N) := \{\varphi \in C^\infty(\mathbb{R}^N) \mid \|\varphi\|_{k,l} < \infty \text{ for all } k,l \in \mathbb{N}_0\}$ is called the space of tempered functions. It is also known as the space of rapidly decreasing/decaying (towards infinity) functions. Note that the definition

$$\mathcal{S}(\mathbb{R}^N) = \left\{\varphi \in C^\infty(\mathbb{R}^N)\ \middle|\ \sup_{x\in\mathbb{R}^N}|x^\alpha\partial^\beta\varphi(x)| < \infty \quad \text{for all multi-indices } \alpha,\beta\right\}$$

is equivalent-

b) The dual space of $\mathcal{S}(\mathbb{R}^N)$ (the space of all distributions on $\mathcal{S}(\mathbb{R}^N)$), denoted by $\mathcal{S}^*(\mathbb{R}^N)$, is called the space of tempered distributions or the space of slowly growing distributions.

c) We set $\mathcal{D}(\mathbb{R}^N) := C^\infty_{\text{comp}}(\mathbb{R}^N)$, with $C^\infty_{\text{comp}}(\mathbb{R}^N)$ as in McLean, W. (2000),.

d) Let N = 1 .By $\mathcal{D}^*_+(\mathbb{R})$, we denote the space of causal distributions, i.e. the space of distributions with support in $[0,\infty)$, and by $\mathcal{S}^*_+(\mathbb{R})$ the space of causal tempered distributions. Using Remark 1 b), the use of the Laplace transform, makes sense for $f \in \mathcal{D}^*_+(\mathbb{R})$ with $e^{-\sigma_0 t}f \in \mathcal{S}^*_+(\mathbb{R})$ for some $\sigma_0 \in \mathbb{R}$.

In this case, i.e. if $f \in \mathcal{D}^*_+(\mathbb{R})$, $\mathcal{L}[f](\omega)$ is holomorphic for $\omega = \mu + i\sigma$ with. $\Im(\omega) = \sigma > \sigma_0$. We thus have the set LT of Laplace transformable distributions given by Tr`eves, F. (1975),

$$LT := \bigcup_{\sigma_0\in\mathbb{R}} LT(\sigma_0)$$

where, for any $\sigma_0 \in \mathbb{R}$,

$$LT(\sigma_0) := \{\ f \in \mathcal{D}^*_+(\mathbb{R})\ \mid\ e^{-\sigma_0 t}f \in \mathcal{S}^*_+(\mathbb{R})\ \}.$$

For any $f \in LT$, we set $\sigma(f) := \inf\{\ \sigma_0 \mid f \in LT(\sigma_0)\ \}$.

Remark 2 (On Definition 1)

There holds $\mathcal{D}(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N)$ and $\mathcal{S}^*(\mathbb{R}^N) \subseteq \mathcal{D}^*(\mathbb{R}^N)$.

Similar definitions for domains $\Omega \subseteq \mathbb{R}^N$ can be found in any textbook on functional analysis. Here we need to extend the definitions above to functions and distributions valued in Banach spaces, as in Tr`eves, F. (1975).

Definition 2 (Generalisation of Definition 1 to Banach Spaces) Let E be a Banach space.

a) Let $\|f\|_{k,l} := \sup_{t\in\mathbb{R}}(t^2+1)^k\sum_{|\alpha|\le l}\|D^\alpha f(\cdot,t)\|_E$ for $k,l \ge 0$ and any multi-index α. Then S(E) $:=$ $\{f(\cdot,t) \in C^\infty(\mathbb{R})$, valued in $E \mid \|f(\cdot,t)\|_{k,l} < \infty$ for all $k,l \in \mathbb{N}_0\}$ is called the space of tempered E-valued functions. Again, $\mathcal{F} : \mathcal{S}(E) \to \mathcal{S}(E)$ is an isomorphism.

b) The dual space of S(E) (the space of all distributions on S(E)), denoted by $\mathcal{S}^*(E)$, is called the space of tempered E-valued distributions or the space of E-valued distributions of slow growth. An equivalent definition can be found in Tr`eves, F. (1975),. Again, $\mathcal{S}^*(E)$ can be identified with a subspace of $\mathcal{D}^*(E)$. For causal distributions, i.e. those with support on $[0,\infty)$, the respective spaces are again denoted by $\mathcal{S}^*_+(E)$ and $\mathcal{D}^*_+(E)$.

c) The set LT(E) of Laplace transformable distributions with values in E is given by

$$LT(E) := \bigcup_{\sigma_0\in\mathbb{R}} LT(\sigma_0, E)$$

where, for any $\sigma_0 \in \mathbb{R}$,

$$LT(\sigma_0, E) := \{\ f(\cdot,t) \in \mathcal{D}^*_+(E)\ \mid\ e^{-\sigma_0 t}f(\cdot,t) \in \mathcal{S}^*_+(E)\ \}.$$

Again, for any $f \in LT(E)$, we set $\sigma(f) := \inf\{\ \sigma_0\ \mid\ f \in LT(\sigma_0, E)\ \}$.

For reference, we state the well known Paley-Wiener Theorem and the Parseval-Plancherel identity. Lemma 1 allows to map results on existence and uniqueness obtained in the frequency domain to the space-time domain, and Lemma 4.2 can be used to deduce mapping properties of time dependent operators from the mapping properties of time independent operators.

Lemma 1 :

Let the E-valued f unction $\hat{f}(\cdot,\omega)=\hat{f}(\cdot,\mu+i\sigma)$ be holomorphic in the half-plane $\mathbb{C}_{\sigma_0}:=\{\omega\in\mathbb{C} \mid \Im(\omega)=\sigma>\sigma_0\}$ for $\sigma_0\in\mathbb{R}$. Then the following conditions are equivalent:

a) There exists a distribution $f\in LT(E)$ such that $\mathcal{L}_t[f(t,\cdot)](\omega,\cdot)=\hat{f}(\omega,\cdot)$.

b) There exists some $\sigma_1>\sigma_0$, some $C>0$ and $k\geq 0$ such that $\|\hat{f}(\cdot,\omega)\|_E\leq C(1+|\omega|)^k$ for all ω with $\Im(\omega)\geq\sigma_1$.

Lemma 2 :

Let E be a Hilbert space, $\omega=\mu+i\sigma$ and $f,g\in LT(E)\cap L^1_{loc}(\mathbb{R},E)$. Then

$$\frac{1}{2\pi}\int_{\mathbb{R}+i\sigma}(\hat{f}(\omega),\hat{g}(\omega))_E\,d\omega=\frac{1}{2\pi}\int_{\mathbb{R}}(\hat{f}(\mu+i\sigma),\hat{g}(\mu+i\sigma))_E\,d\mu=\int_{\mathbb{R}}e^{-2\sigma t}(f(t),g(t))_E\,dt.$$

With $\sigma=0$, we obtain

$$\frac{1}{2\pi}\int_{\mathbb{R}}(\tilde{f}(\mu),\tilde{g}(\mu))_E\,d\mu=\int_{\mathbb{R}}(f(t),g(t))_E\,dt$$

for the Fourier transform.

As in A. Bamberger and T. Ha-Duong (1986),, we further define an operator Λ^s implicitly by

$$\Lambda^s[f]:=\mathcal{L}^{-1}\left[(-i\omega)^s\hat{f}(\omega)\right] \tag{4}$$

for any $s\in\mathbb{R}$. For s=k with $k\in\mathbb{N}$, this is the k-th temporal derivative of f, whereas for s = -k with $k\in\mathbb{N}$, this is the k-th temporal anti-derivative of f.

ANALYSIS IN THE LAPLACE DOMAIN

As we mentioned in the introduction to this section, we use the classical approach for the analysis of the time domain boundary layer potentials and the corresponding time domain boundary integral operators that was introduced by Bamberger and Ha-Duong (1986). This means that we analyse the potentials and operators in the Laplace domain first, after mapping the original problem (P) to the Laplace domain by means of the Laplace transform.

Reviews on this method have been published in the form of research papers, such as the ones by Ha-Duong (2003) or Laliena and Sayas (2009) [103], or in the form of lecture notes, such as the ones by Becache (1994) or, mast recently, Sayas (2011).

Let us recall how the transient problem (P) is related to the time harmonic Helmholtz problem first. Assume that u solves (P), and let u:= C_t[u]. Then u solves the exterior Helmholtz boundary value problem

$$(\mathrm{HH})\begin{cases}\text{Find } u:\Omega\to\mathbb{R} \text{ such that} & \\ \qquad \Delta u+\omega^2 u = 0 \text{ in } \Omega & \\ \qquad \mathcal{B}[u] = \hat{g} \text{ on } \Gamma & \\ \qquad \lim_{R\to\infty}\int_{\partial\mathbb{B}_R(0)}|\nabla u\cdot n_x-i\omega u|^2\,ds_x = 0 & \\ \text{where } \hat{g}:=\mathcal{L}_t[g]:\Gamma\to\mathbb{R}, \text{ with } g \text{ as in (P).} & \end{cases}$$

Equation 5a) is known as the homogeneous *Helmholtz equation,* while condition (5c) is called the *Sommerfeld radiation condition.*

We note that there are at least two different versions of the Sommerfeld radiation condition by which (5c) could be replaced. For the definition of the boundary layer potentials and integral operators, we state the fundamental solutions of the Helmholtz equation in Lemma 3.

Lemma 3 (Fundamental Solutions of the Helmholtz Equation)

The fundamental solution of the Helmholtz equation (5a)

$$(1D):\quad G_\omega(x,y) = G_\omega(|x-y|) = -\frac{1}{2i\omega}e^{i\omega|x-y|} \tag{6}$$

$$(2D):\quad G_\omega(x,y) = G_\omega(|x-y|) = \frac{i}{4}H_0^{(1)}(\omega|x-y|) \tag{7}$$

$$(3D):\quad G_\omega(x,y) = G_\omega(|x-y|) = \frac{1}{4\pi}\frac{e^{i\omega|x-y|}}{|x-y|} \tag{8}$$

with $\omega\in\mathbb{C}$, where $H_0^{(1)}$ denotes the Hankel function of order zero of the first, kind.

The boundary potentials for the Helmholtz Problem (HH) are defined similarly to the time domain boundary layer potentials given in Definition 3.

Definition 4 (Boundary Layer Potentials for the Helmholtz Problem)

Let $x\in(\mathbb{R}^n\setminus\Gamma)$. For appropriate densities $p,\varphi:\Gamma\to\mathbb{R}$, the Helmholtz Single Layer potential is given by

$$S_\omega[p](x)=\int_\Gamma G_\omega(|x-y|)\;p(y)\,ds_y \tag{9}$$

and the Helmholtz Double Layer potential by

$$D_\omega[\varphi](x)=\int_\Gamma n_y\cdot\nabla_x G_\omega(|x-y|)\;\varphi(y)\,ds_y. \tag{10}$$

As in Definition. we do not specify the regularity of p and φ here. We conduct an explicit analysis that takes the dependency on the wave number ω into account. We further refer to Costabel, M. (1988) for mapping properties of arbitrary elliptic boundary integral operators.

The corresponding boundary integral operators $V_\omega, K_\omega, K'_\omega$ and W_ω are defined analogously to Definition. Note that the Helmholtz boundary potentials and operators are the Laplace transforms of their transient counterparts, e.g. $S_\omega[\hat{p}] = \mathcal{L}_t[S[p]]$.

In the rest of this study, we restrict ourselves to the three dimensional case, even though we conduct our studies on a posteriori error estimates and our numerical experiments in two space dimensions. This distinction can be justified by the fact that the fundamental solution of the two dimensional Helmholtz equation is much more complicated than its three dimensional counterpart, and therefore dealing with it would lead to many additional technical difficulties. Other authors have followed these two different routes in theory and practice for the same reason. The new generalised mapping properties studied, however, also hold in two space dimensions.

Note that we omit the hat while working in the Laplace domain and write ***u*** instead of $\hat{u}$ if its correct meaning is clear and there is no danger of confusion.

An Equivalent Norm in Sobolev Spaces-

For some domain Ω, the energy of ***u*** in Ω is given

$$E_\Omega[u](t) := \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 + \dot{u}(x,t)^2\, dx. \tag{11}$$

By the Parseval-Plancherel identity (Lemma 2),

$$\int_{\mathbb{R}} e^{-2\sigma t} E_\Omega[u](t)\, dt = \frac{1}{4\pi}\int_{\mathbb{R}+i\sigma}\int_\Omega |\nabla \hat{u}(x)|^2 + (\omega \hat{u}(x))^2\, dx\, d\omega. \tag{12}$$

This relation motivates the definition of the following energy-related norms.

Recall the definition of the usual $H^s(\mathbb{R}^N)$-norm

$$\|u\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (1+|\xi|^2)^s |\tilde{u}(\xi)|^2\, d\xi. \tag{13}$$

Now, to guarantee $\omega \neq 0$, let $\Im(\omega) > \sigma_0$ for some $\sigma_0 \in \mathbb{R}_{>0}$. Then the norm

$$\|u\|^2_{s,\omega,\mathbb{R}^N} = \int_{\mathbb{R}^N} (|\omega|^2 + |\xi|^2)^s |\tilde{u}(\xi)|^2\, d\xi \tag{14}$$

is equivalent to the $\|\cdot\|_{H^s(\mathbb{R}^N)}$-norm. We often refer to thesenorms as the wavenumber-dependent norms or $|\omega|$ -dependent noims.

The norms $\|\cdot\|^2_{s,\omega,\Omega}$ and $\|\cdot\|^2_{s,\omega,\Gamma}$ are defined analogously by using an atlas. We write $\|\cdot\|_{s,\omega,\mathcal{O}}$ and $H^s(\mathcal{O})$ when we deal with statements and results that hold for all $\mathcal{O} \in \{\mathbb{R}^N, \Omega, \Gamma\}$.

Remark (Equivalence Estimates for the Classical and |u;|-Dependent Norms)

Note that the equivalence of the norms Ls $|\omega|$ -dependent. We have, for $s \geq 0$, $(|\omega|^2 + |\xi|^2)^s \geq (\sigma_0^2 + |\xi|^2)^s \geq \min\{1, \sigma_0^{2s}\}(1+|\xi|^2)^s$ but only

$$(|\omega|^2 + |\xi|^2)^s \leq \left(|\omega|^2 + \frac{1}{\sigma_0^2}|\omega|^2|\xi|^2\right)^s \leq \max\left\{1, \frac{1}{\sigma_0^{2s}}\right\}|\omega|^{2s}(1+|\xi|^2)^s$$

and hence, in summary,

$$\min\{1, \sigma_0^{2s}\}(1+|\xi|^2)^s \leq (|\omega|^2 + |\xi|^2)^s \leq \max\left\{1, \frac{1}{\sigma_0^{2s}}\right\}|\omega|^{2s}(1+|\xi|^2)^s. \tag{15}$$

Correspondingly, for $s \leq 0$,

$$\min\{1, \sigma_0^{2s}\}|\omega|^{2s}(1+|\xi|^2)^s \leq (|\omega|^2 + |\xi|^2)^s \leq \max\left\{1, \frac{1}{\sigma_0^{2s}}\right\}(1+|\xi|^2)^s. \tag{16}$$

(15) and (16) can be summarised as

$$C_1(\sigma_0)\left(|\omega|^{2s}\right)^{H(-s)}(1+|\xi|^2)^s \leq (|\omega|^2 + |\xi|^2)^s \leq C_2(\sigma_0)\left(|\omega|^{2s}\right)^{H(s)}(1+|\xi|^2)^s \tag{17}$$

for any $s \in \mathbb{R}$. Norm equivalences of the type

$$C(\sigma_0)\|u\|_{H^1(\mathcal{O})} \leq \|u\|_{1,\omega,\mathcal{O}} \leq \tilde{C}(\sigma_0)|\omega|\|u\|_{H^1(\mathcal{O})} \tag{18}$$

for $\mathcal{O} \in \{\mathbb{R}^N, \Omega, \Gamma\}$ can be concluded from (17).

Example (Norm Equivalences)

a) For s= 1

$$\|u\|^2_{1,\omega,\mathbb{R}^N} = |\omega|^2 \int_{\mathbb{R}^N} |\tilde{u}(\xi)|^2\, d\xi + \int_{\mathbb{R}^N} |\nabla \tilde{u}(\xi)|^2\, d\xi = |\omega|^2\|u\|^2_{L^2(\mathbb{R}^N)} + |u|^2_{H^1(\mathbb{R}^N)}.$$

Similarly, for s = 2,

$$\|u\|^2_{2,\omega,\mathbb{R}^N} = |\omega|^4\|u\|^2_{L^2(\mathbb{R}^N)} + 2|\omega|^2|u|^2_{H^1(\mathbb{R}^N)} + |u|^2_{H^2(\mathbb{R}^N)}.$$

b) Since, for $a, b \geq 0, \sqrt{\tfrac{1}{2}}(a+b) \leq \sqrt{a^2+b^2} \leq a+b,$ there holds $\sqrt{\tfrac{1}{2}}\left(|\omega|\|u\|^2_{L^2(\mathbb{R}^N)} + |u|^2_{H^{1/2}(\mathbb{R}^N)}\right) \leq \|u\|^2_{1/2,\omega,\mathbb{R}^N} \leq |\omega|\|u\|^2_{L^2(\mathbb{R}^N)} + |u|^2_{H^{1/2}(\mathbb{R}^N)}$ and therefore equivalence of these two norms.

Due to the $|\omega|$-dependence of the equivalence estimates illustrated in Remark, we cannot deduce, for instance, the trace theorem for the $|\omega|$-dependent norms directly from the results for the standard norms without any dosses' regarding powers of $|\omega|$

The Trace Theorem for the $|\omega|$-Dependent Norms -

The proofs of the following results are all similar to the ones of the original results given in McLean, W. (2000). For the particular case ***s*** = 1, a result similar to Lemma 4 below, based on the proof of for the standard norms. These are also collected in Lemma. We give outlines of the proofs here for completeness and to demonstrate the (minor) differences to the original proofs. The term 'generalisation' Ls appropriate in the sense that we obtain the classical theorems again for $|\omega| = 1.$

Lemma 4 :

a) For $s > \frac{1}{2},$ the trace operator $\gamma : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^{n-1}),$ given by $\gamma[u](x) = \gamma[u](x', x_n) := u(x', 0)$ has a unique extension to a bounded linear operator $\gamma : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$

where the continuity constant with respect to the $|\omega|$-dependent noim $\| \cdot \|_{s,\omega,\mathcal{O}}$ depends on but not on $|\omega|$, i.e. there exists a $|\omega|$-independent constant $C = C(s) > 0$ such that

$$\|\gamma[u]\|_{s-1/2,\omega,\mathbb{R}^{n-1}} \leq C\|u\|_{s,\omega,\mathbb{R}^n} \quad (19)$$

for $u \in H^s(\mathbb{R}^n).$

b) Let Ω be $a\ C^{k,1}$ domain and $\frac{1}{2} < s \leq k+1.$ Thenthe trace operator $\gamma : \mathcal{D}(\bar{\Omega}) \to \mathcal{D}(\Gamma),$ defined by $\gamma[u] := u|_\Gamma,$ has an extension to a linear bounded operator $\gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ where the continuity constant with respect to the $|\omega|$-dependent norms $\| \cdot \|_{s,\omega,\mathcal{O}},\ \mathcal{O} \in \{\Omega, \Gamma\},$ depends on s, σ_0 and Γ but not. on $|\omega|$ in the sense of part a).

Proof

Part b) is a consequence of part a), using a technique called 'flattening of the boundary' described in the proof of McLean, W. (2000), It is thus enough to show part a).

We write $x = (x', x_n)$ and $\xi = (\xi', \xi_n)$ for $x, \xi \in \mathbb{R}^n.$ Since

$$\tilde{u}(\xi) := \mathcal{F}_x[u](\xi) = \mathcal{F}_{x_n}\left[\mathcal{F}_{x'}[u](\xi', x_n)\right](\xi', \xi_n)$$

there holds, by the definition of the inverse Fourier transform,

$$\mathcal{F}_{x_n}^{-1}[\tilde{u}](\xi', x_n) = \int_{\mathbb{R}} e^{i 2\pi \xi_n x_n} \tilde{u}(\xi', \xi_n)\, d\xi_n = \mathcal{F}_{x'}[u](\xi', x_n).$$

By the definition, $\gamma[u](x) = \gamma[u](x') = u(x', 0)$ and thus

$$\mathcal{F}_{x'}[\gamma[u]](\xi') = \int_{\mathbb{R}} \tilde{u}(\xi', \xi_n)\, d\xi_n = \int_{\mathbb{R}} \left(|\omega|^2 + |\xi|^2\right)^{-s/2} \left(|\omega|^2 + |\xi|^2\right)^{s/2} \tilde{u}(\xi', \xi_n)\, d\xi_n. \quad (20)$$

By the Cauchy-Schwarz inequality for integrals, we obtain

$$\left|\mathcal{F}_{x'}[\gamma[u]](\xi')\right|^2 \leq M_s^\omega(\xi') \int_{\mathbb{R}} \left(|\omega|^2 + |\xi|^2\right)^s \left|\tilde{u}(\xi', \xi_n)\right|^2 d\xi_n. \quad (21)$$

where

$$M_s^\omega(\xi') := \int_{\mathbb{R}} \left(|\omega|^2 + |\xi|^2\right)^{-s} d\xi_n = \left(|\omega|^2 + |\xi'|^2\right)^{1/2-s} \int_{\mathbb{R}} (t^2+1)^{-s}\, dt \quad (22)$$

via the substitution $t(\xi_n) := \xi_n \left(|\omega|^2 + |\xi'|^2\right)^{-1/2}$ that gives $d\xi_n = \left(|\omega|^2 + |\xi'|^2\right)^{1/2}$ ***dt*** and $|\omega|^2 + |\xi|^2 =$ $(1+t^2)\left(|\omega|^2 + |\xi'|^2\right).$ By McLean, W. (2000),, there holds, for $s > 1/2,$ $C_s := \int_{\mathbb{R}} (t^2+1)^{-s}\, dt < \infty$ where the constant C_s is obviously $|\omega|$-independent. Combining (21) and (22), we obtain

$$\left(|\omega|^2 + |\xi'|^2\right)^{s-1/2} \left|\mathcal{F}_{x'}[\gamma[u]](\xi')\right|^2 \leq C_s \int_{\mathbb{R}} \left(|\omega|^2 + |\xi|^2\right)^s \left|\tilde{u}(\xi', \xi_n)\right|^2 d\xi_n. \quad (23)$$

Taking the integral $\int_{\mathbb{R}^{n-1}} d\xi'$ in (23) proves the claimed result via the definitions of the respective norms.

Since a Lipschitz boundary is $C^{0,1},$ the result above can only be applied for $\frac{1}{2} < s \leq 1$ in this case.

In fact it can be extended to the range $1 < s < \frac{3}{2},$ as the following result shows. The proof is again similar to the original ones of McLean, W. (2000),.

Lemma 5 :

Let Ω be a Lipschitz domain and $\frac{1}{2} < s < \frac{3}{2}.$ Then the trace opemtor defined in Lemma 4 b) is bounded independently of $|\omega|$ as an operator mapping $H^s(\Omega)$ to $H^{s-1/2}(\Gamma)$ in the same way as in Lemma 4 b), i.e. with a continuity constant that depends only on s, σ_0 and $\Gamma.$

The following result is an immediate consequence of Lemma 4.5 and the fact that the norms of a linear operator and of its dual operator are equal; see, for instance, McLean, W. (2000).

Corollary 1:

Let Ω be a Lipschitz domain and $\frac{1}{2} < s < \frac{3}{2}.$ Then the adjoint operator γ^* to the trace operator γ defined in Lemma 4 b) is bounded independently of ω as an operator mapping $\left(H^{s-1/2}(\Gamma)\right)^* = H^{1/2-s}(\Gamma)$ to $\left(H^s(\Omega)\right)^* = H^{-s}(\Omega)$ in the same way as in Lemma 5.

Proof (of Lemma 5)

First we define an anisotropic Sobolev space E_ω^s via the norm

$$\|u\|_{E_\omega^s}^2 := \int_{\mathbb{R}} |\xi_n|^{2s} \|\mathcal{F}_{x_n}[u(\cdot, x_n)](\cdot, \xi_n)\|_{L^2(\mathbb{R}^{n-1})}^2 + |\xi_n|^{2(s-1)} \|\mathcal{F}_{x_n}[u(\cdot, x_n)](\cdot, \xi_n)\|_{1,\omega,\mathbb{R}^{n-1}}^2\, d\xi_n.$$

By the definitions of the norms, there holds

$$\|u\|^2_{E^s_\omega} = \int_{\mathbb{R}^n} |\xi_n|^{2(s-1)} \left(|\omega|^2 + |\xi|^2\right) |\tilde{u}(\xi)|^2 \, d\xi_n$$

where, as in the proof of Lemma 4, $\tilde{u} := \mathcal{F}_x[u]$. Writing $x = (x', x_n)$ and $\xi = (\xi', \xi_n)$ for $x, \xi \in \mathbb{R}^n$ again, we define

$$u_\zeta(x) := u(x', \zeta(x') + x_n)$$

where ζ Ls the Lipschitz-continuous function whose graph is Γ. Then, by the definition,

$$\|\gamma[u]\|_{s-1/2,\omega,\Gamma} = \|u_\zeta(\cdot,0)\|_{s-1/2,\omega,\mathbb{R}^{n-1}}. \tag{24}$$

Analogously to the proof of McLean, W. (2000) theorem, there hold the inequalities

$$\|u_\zeta\|_{E^s_\omega} \le C\|u\|_{E^s_\omega} \tag{25}$$

and

$$\|u\|_{E^s_\omega} \le \|u\|_{s,\omega,\mathbb{R}^n} \tag{26}$$

where $C = C(\Omega)$ Ls $|\omega|$-independent. Further, by the definition and (20), and by the Cauchy-Schwarz inequality for integrals,

$$\begin{aligned}\|u(\cdot,0)\|^2_{s-1/2,\omega,\mathbb{R}^{n-1}} &= \int_{\mathbb{R}^{n-1}} \left(|\omega|^2 + |\xi'|^2\right)^{s-1/2} \left|\int_{\mathbb{R}} \tilde{u}(\xi)\, d\xi_n\right|^2 d\xi' \\ &\le \int_{\mathbb{R}^{n-1}} \left(|\omega|^2 + |\xi'|^2\right)^{s-1/2} \left(\int_{\mathbb{R}} |\xi_n|^{2(1-s)} \left(|\omega|^2 + |\xi|^2\right)^{-1} d\xi_n\right) \\ &\qquad \int_{\mathbb{R}} |\xi_n|^{2(s-1)} \left(|\omega|^2 + |\xi|^2\right) |\tilde{u}(\xi)|^2 \, d\xi_n \; d\xi'.\end{aligned}$$

Using the same substitution as in the proof of Lemma 4, we obtain

$$\begin{aligned}&\int_{\mathbb{R}} |\xi_n|^{2(1-s)} \left(|\omega|^2 + |\xi|^2\right)^{-1} d\xi_n \\ &= \int_{\mathbb{R}} t^{2(1-s)} \left(|\omega|^2 + |\xi'|^2\right)^{1-s} (t^2+1)^{-1} \left(|\omega|^2 + |\xi'|^2\right)^{-1} \left(|\omega|^2 + |\xi'|^2\right)^{1/2} dt \\ &= \left(|\omega|^2 + |\xi'|^2\right)^{1/2-s} \int_{\mathbb{R}} (t^2+1)^{-1}\, t^{2(1-s)}\, dt =: \left(|\omega|^2 + |\xi'|^2\right)^{1/2-s} C_s\end{aligned}$$

where the constant $C_s < \infty$ is obviously $|\omega|$-independent . Thus

$$\|u(\cdot,0)\|^2_{s-1/2,\omega,\mathbb{R}^{n-1}} \tag{27}$$

$$\begin{aligned}&\le \int_{\mathbb{R}^{n-1}} \left(|\omega|^2 + |\xi'|^2\right)^{s-1/2} C_s \left(|\omega|^2 + |\xi'|^2\right)^{1/2-s} \int_{\mathbb{R}} |\xi_n|^{2(s-1)} \left(|\omega|^2 + |\xi|^2\right) |\tilde{u}(\xi)|^2 \, d\xi_n \\ &= C_s \int_{\mathbb{R}^n} |\xi_n|^{2(s-1)} \left(|\omega|^2 + |\xi|^2\right) |\tilde{u}(\xi)|^2 \, d\xi \;=\; C_s \|u\|^2_{E^s_\omega}.\end{aligned}$$

Combining estimates (24), (27), (25) and (26), we obtain

$$\|\gamma[u]\|_{s-1/2,\omega,\Gamma} = \|u_\zeta(\cdot,0)\|_{s-1/2,\omega,\mathbb{R}^{n-1}} \le \sqrt{C_s}\|u_\zeta\|_{E^s_\omega} \le \sqrt{C_s}C\|u\|_{E^s_\omega} \le \sqrt{C_s}C\|u\|_{s,\omega,\mathbb{R}^n}$$

which proves the claim.

The next result is about the inverse to γ, the so-called extension operator. As before, the proofs follow the respective ones given in McLean, W. (2000), closely.

Lemma 6 :

a) For each $j \in \mathbb{Z}_{\ge 0}$, there exists an $|\omega|$-dependent bounded linear operator

$$\eta_j^\omega : H^{s-j-1/2}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^n)$$

where the continuity constant with respect to the $|\omega|$-dependent norm $\|\cdot\|_{s,\omega,\mathbb{R}^N}$ depends on s, but not on $|\omega|$ in the sense of Lemma 7), i.e. there exists a $|\omega|$-independent constant $C = C(s) > 0$ such that

$$\|\eta_j^\omega[u]\|_{s,\omega,\mathbb{R}^n} \le C\|u\|_{s-j-1/2,\omega,\mathbb{R}^{n-1}} \tag{28}$$

for $u \in H^{s-j-1/2}(\mathbb{R}^{n-1})$.

b) Let Ω be a $C^{k,1}$ domain and $\frac{1}{2} < s \le k+1$. Then there exists a $|\omega|$-dependent bounded linear operator

$$Z^\omega_\Omega : H^{s-1/2}(\Gamma) \to H^s(\Omega)$$

which is a right inverse to γ, i.e. $(\gamma \circ Z^\omega_\Omega)[\varphi] \equiv \varphi$ for $\varphi \in H^{s-1/2}(\Gamma)$. The continuity constant with res-pect to the $|\omega|$-dependent norms $\|\cdot\|_{s,\omega,\mathcal{O}}$, $\mathcal{O} \in \{\Omega,\Gamma\}$, depends on s, σ_0 and Γ, but not on $|\omega|$ in the sense of part a). Z^ω_Ω is sometimes called the extension operator to φ.

Proof

Part b) is a consequence of part a); taking $Z^\omega_\Omega = \eta^\omega_0$ and using the same technique as in the proof of Lemma 7 b). It is thus enough to show part a).

Take $\theta_j \in \mathcal{D}(\mathbb{R})$ such that $\theta_j(y) = \frac{y^j}{j!}$ for $|y| \le 1$. The operator η_j^ω is defined by, for $x \in \mathbb{R}^n$,

$$\eta_j^\omega[u](x) := \int_{\mathbb{R}^{n-1}} \frac{\tilde{u}(\xi')\,\theta_j\left(\left(|\omega|^2 + |\xi'|^2\right)^{1/2} x_n\right)}{\left(|\omega|^2 + |\xi'|^2\right)^{j/2}} e^{i2\pi\xi' \cdot x'}\, d\xi' \tag{29}$$

where, in this case. $\tilde{u} = \mathcal{F}_{x'}[u]$.

Then

$$\begin{aligned}&\mathcal{F}_{x_n}\left[\eta_j^\omega[u]\right](x',\xi_n) = \int_{\mathbb{R}} \eta_j^\omega[u](x',x_n)\, e^{-i2\pi x_n \xi_n}\, dx_n \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \theta_j\left(\left(|\omega|^2 + |\xi'|^2\right)^{1/2} x_n\right) e^{-i2\pi x_n \xi_n}\, dx_n \, \frac{\tilde{u}(\xi')}{\left(|\omega|^2 + |\xi'|^2\right)^{j/2}} e^{i2\pi\xi' \cdot x'}\, d\xi' \\ &= \mathcal{F}^{-1}_{\xi'}\left[\frac{\tilde{u}(\xi')}{\left(|\omega|^2 + |\xi'|^2\right)^{j/2}} \int_{\mathbb{R}} \theta_j\left(\left(|\omega|^2 + |\xi'|^2\right)^{1/2} x_n\right) e^{-i2\pi x_n \xi_n}\, dx_n\right](x',\xi_n)\end{aligned}$$

and thus

$$\begin{aligned}\mathcal{F}_x\left[\eta_j^\omega[u]\right](\xi) &= \mathcal{F}_{x'}\left[\mathcal{F}_{x_n}\left[\eta_j^\omega[u]\right](x',\xi_n)\right](\xi',\xi_n) \\ &= \frac{\tilde{u}(\xi')}{\left(|\omega|^2 + |\xi'|^2\right)^{j/2}} \int_{\mathbb{R}} \theta_j\left(\left(|\omega|^2 + |\xi'|^2\right)^{1/2} x_n\right) e^{-i2\pi x_n \xi_n}\, dx_n.\end{aligned}$$

Substituting $y(x_n) := \left(|\omega|^2 + |\xi'|^2\right)^{1/2} x_n$, we obtain, with $dx_n = \left(|\omega|^2 + |\xi'|^2\right)^{-1/2} dy$,

$$\int_{\mathbb{R}} \theta_j \left((|\omega|^2 + |\xi'|^2)^{1/2} x_n \right) e^{-i2\pi x_n \xi_n} \, dx_n$$
$$= \left((|\omega|^2 + |\xi'|^2 \right)^{-1/2} \int_{\mathbb{R}} \theta_j(y) \; e^{-i2\pi y \left(\xi_n (|\omega|^2 + |\xi'|^2)^{-1/2} \right)} \, dy$$
$$= \left(|\omega|^2 + |\xi'|^2 \right)^{-1/2} \mathcal{F}_{x_n}[\theta_j] \left(\xi_n \left(|\omega|^2 + |\xi'|^2 \right)^{-1/2} \right)$$

and hence $\mathcal{F}_x[\eta_j^\omega[u]](\xi) = \tilde{u}(\xi') \left(|\omega|^2 + |\xi'|^2 \right)^{-(j+1)/2} \mathcal{F}_{x_n}[\theta_j] \left(\xi_n \left(|\omega|^2 + |\xi'|^2 \right)^{-1/2} \right).$
Now

$$\|\eta_j^\omega[u]\|^2_{s,\omega,\mathbb{R}^n} = \int_{\mathbb{R}^n} (|\omega|^2 + |\xi|^2)^s \left| \mathcal{F}_x[\eta_j^\omega[u]](\xi) \right|^2 d\xi$$
$$= \int_{\mathbb{R}^{n-1}} \frac{|\tilde{u}(\xi')|^2}{(|\omega|^2 + |\xi'|^2)^{j+1}} \int_{\mathbb{R}} (|\omega|^2 + |\xi|^2)^s \left| \mathcal{F}_{x_n}[\theta_j] \left(\xi_n \left(|\omega|^2 + |\xi'|^2 \right)^{-1/2} \right) \right|^2 d\xi_n \, d\xi'.$$

Using the same substitution as in the proof of Lemma 4, we obtain

$$\|\eta_j^\omega[u]\|^2_{s,\omega,\mathbb{R}^n}$$
$$= \int_{\mathbb{R}^{n-1}} \frac{|\tilde{u}(\xi')|^2}{(|\omega|^2 + |\xi'|^2)^{j+1}} \int_{\mathbb{R}} (|\omega|^2 + |\xi'|^2)^s (1+t^2)^s (|\omega|^2 + |\xi'|^2)^{1/2} |\mathcal{F}_{x_n}[\theta_j](t)|^2 \, dt \, d\xi'$$
$$= \int_{\mathbb{R}^{n-1}} |\tilde{u}(\xi')|^2 \left(|\omega|^2 + |\xi'|^2 \right)^{s-j-1/2} \int_{\mathbb{R}} (1+t^2)^s |\mathcal{F}_{x_n}[\theta_j](t)|^2 \, dt \, d\xi'$$

where the integral $\int_{\mathbb{R}} (1+t^2)^s |\mathcal{F}_{x_n}[\theta_j](t)|^2 \, dt =: C_s < \infty$ is bounded independently of $|\omega|$ for all $s \in \mathbb{R}$ and thus finally

$$\|\eta_j^\omega[u]\|^2_{s,\omega,\mathbb{R}^n} = C_s \|\eta_j^\omega[u]\|^2_{s-j-1/2,\omega,\mathbb{R}^{n-1}}.$$

This completes the proof.

A Brief Remark on Interpolation -

We close our observations on the $|\omega|$-dependent norms with some brief remarks on interpolation. We use the notation $[X,Y]_\theta$ with $\theta \in (0,1)$ for interpolation spaces here, where ***X***, ***Y*** are Banach spaces with $X \subseteq Y$. Without providing any technical details, we first cite a result for future reference.

Lemma 7 :

For any $u \in X \subseteq Y,$ there holds $\|u\|_{[X,Y]_\theta} \le \|u\|_X^{1/2-\theta} \|u\|_Y^{1/2+\theta}.$

We now return to our specific setup. It is well established that $[H^{s_0}(\mathcal{O}), H^{s_1}(\mathcal{O})]_\theta = H^s(\mathcal{O})$ for $s = (1-\theta)s_0 + \theta s_1,\ \theta \in (0,1)$ and $\mathcal{O} \in \{\mathbb{R}^N, \Omega, \Gamma\};$. Inwhat follows, we need:

Theorem 1 (Interpolation for Sobolev Spaces $H^s(\mathcal{O})$

Let the linear operator A be bounded as $A : H^{s_0}(\mathcal{O}) \to H^{t_0}(\mathcal{O})$ and as $A : H^{s_1}(\mathcal{O}) \to H^{t_1}(\mathcal{O})$ with $\|A[u]\|_{H^{t_j}(\mathcal{O})} \le M_j \|u\|_{H^{s_j}(\mathcal{O})}$ for j = 0,1. Then

$$\|A[u]\|_{H^t(\mathcal{O})} \le M_0^{1-\theta} M_1^\theta \|u\|_{H^s(\mathcal{O})} \qquad (30)$$

for $s = (1-\theta)s_0 + \theta s_1,\ t = (1-\theta)t_0 + \theta t_1$ and $\theta \in (0,1),$ i.e. $A : H^s(\mathcal{O}) \to H^t(\mathcal{O})$ is also a bounded map for these numbers s,t.

Theorem 1 is stated in McLean, W. (2000),in general, but it is cited here for the special case of Sobolev spaces. In what follows, it is shown that Theorem 4.1 still holds without any additional $|\omega|$-dependent factors in (30) if the wave number dependent norms are used. To do this, we follow the proofs in McLean, W. (2000),. Let

$$K(t,u;(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n))) := \inf_{\substack{u = u_0 + u_1 \\ u_0 \in H^{s_0}(\mathbb{R}^n),\, u_1 \in H^{s_1}(\mathbb{R}^n)}} \left(\|u_0\|^2_{H^{s_0}(\mathbb{R}^n)} + t^2 \|u_1\|^2_{H^{s_1}(\mathbb{R}^n)} \right)$$

for $t > 0$ and $u \in H^{s_0}(\mathbb{R}^n) + H^{s_1}(\mathbb{R}^n).$ It is shown in McLean, W. (2000),] that $K(t,u;(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n))) = \int_{\mathbb{R}^n} (1+|\xi|^2)^{s_0} (f(a(\xi)t))^2 |\tilde{u}(\xi)|^2 \, d\xi$

with $a(\xi) = (1+|\xi|^2)^{(s_1-s_0)/2}$ and $f(t) = \frac{t}{\sqrt{1+t^2}},$ and.

$$\|K(\cdot,u;(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n)))\|_{\theta,2} = \left(\frac{\pi}{2\sin(\pi\theta)} \right)^{1/2} \|u\|_{H^s(\mathbb{R}^n)} =: N_{\theta,2} \|u\|_{H^s(\mathbb{R}^n)}$$

where $\|\cdot\|_{\theta,2}$ is a weighted L_2 norm,

$$\|f\|^2_{\theta,2} := \int_{\mathbb{R}_{\ge 0}} \frac{|t^{-\theta} f(t)|^2}{t} \, dt.$$

Note that $\|u\|_{K_{\theta,2}(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n))} = N_{\theta,2} \|K(\cdot,u;(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n)))\|_{\theta,2}$ is the norm of the interpolation space $[H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n)]_\theta = K_{\theta,2}((H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n)))$ that corresponds to θ.

Repeating the proof of McLean, W. (2000),line by line, we find that, for

$$K_\omega(t,u;(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n))) := \inf_{\substack{u = u_0 + u_1 \\ u_0 \in H^{s_0}(\mathbb{R}^n),\, u_1 \in H^{s_1}(\mathbb{R}^n)}} \left(\|u_0\|^2_{s_0,\omega,\mathbb{R}^n} + t^2 \|u_1\|^2_{s_1,\omega,\mathbb{R}^n} \right)$$

there holds

$$K_\omega(t,u;(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n))) = \int_{\mathbb{R}^n} (|\omega|^2 + |\xi|^2)^{s_0} (f(a_\omega(\xi)t))^2 |\tilde{u}(\xi)|^2 \, d\xi$$

with $a(\xi) = (|\omega|^2 + |\xi|^2)^{(s_1-s_0)/2}$ and $\|K_\omega(\cdot,u;(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n)))\|_{\theta,2} = N_{\theta,2} \|u\|_{s,\omega,\mathbb{R}^n}.$

We can thus use the interpolation result Theorem in the usual way for the $|\omega|$-dependent norms, without any additional factors of $|\omega|$ appearing in the interpolation estimates.

MAPPING PROPERTIES OF THE BOUNDARY POTENTIALS AND BOUNDARY INTEGRAL OPERATORS FOR THE HELMHOLTZ PROBLEM

In this section, we deal with the mapping properties of the boundary layer potentials given in Definition. and of the corresponding boundary integral operators for the Helmholtz problem. The bounds are explicit with respect to the wave number cj, which is going to prove to be important in the next section, where we consider the mapping properties of the time domain boundary layer potentials and the corresponding time domain boundary integral operators.

We first collect estimates with respect to the natural (or energy) norms that have been proven in the literature. By natural norms, we mean the spaces $H^1(\Omega)$, $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ equipped with either the classical norms or with the $|\omega|$-dependent norms introduced. We then use Costabels technique to generalise these results to a wider range of Sobolev spaces. The mapping properties themselves are well known; the new contribution is the explicitness in $|\omega|$

A Review on Estimates with respect to the Natural Norms-

We state the mapping properties in the classical Sobolev norms and in the equivalent $|\omega|$ dependent norms. The differences we observe are merely a result of the differences in part b) and c) of the following lemma. In the estimates, ω is the argument of the Laplace transform. The differences in the estimates with respect to this variable are crucial, as their powers correspond to the orders of the time derivatives in the space-time estimates via the Parseval-Plancherel identity.

Lemma 8 (Trace Theorem; Trace Extension Lemma)

Let Ω be a Lipschitz domain. Then the following results hold.

a) Trace Theorem

(i) For $u \in H^1(\Omega)$, with $C = C(\Gamma)$,

$$\|\gamma[u]\|_{H^{1/2}(\Gamma)} \le C\|u\|_{H^1(\Omega)}.$$

(ii) For $u \in H^1(\Omega)$, with $C = C(\Gamma, \sigma_0)$,

$$\|\gamma[u]\|_{1/2,\omega,\Gamma} \le C\|u\|_{1,\omega,\Omega}.$$

b) Trace Extension Lemma

(i) For $\varphi \in H^{1/2}(\Gamma)$, them exists an extension $u = \mathcal{R}[\varphi]$ into Ω, for which.

$$\|u\|_{1,\omega,\Omega} \le C|\omega|^{1/2}\|\varphi\|_{H^{1/2}(\Gamma)}$$

with. $C = C(\Gamma)$.

(i) For $\varphi \in H^{1/2}(\Gamma)$, there exists an extension $u = \mathcal{R}[\varphi]$ into Ω, for which $\|u\|_{1,\omega,\Omega} \le C\|\varphi\|_{1/2,\omega,\Gamma}$ with. $C = C(\Gamma, \sigma_0)$.

c) Bound for the normal derivative

(i) Let u solve the homogeneous Helmholtz equation (4.5a). Then, with. $C = C(\Gamma, \sigma_0)$,

$$\left\|\frac{\partial u}{\partial n}\right\|_{H^{-1/2}(\Gamma)} \le C|\omega|^{1/2}\|u\|_{1,\omega,\Omega}.$$

A similar result is given in Melenk, J. M. (2010), but there the factor on the right hand side is $|\omega|$ instead of $|\omega|^{1/2}$.

(i) Let u solve the homogeneous Helmholtz equation (5a). Then, with $C = C(\Gamma, \sigma_0)$,

$$\left\|\frac{\partial u}{\partial n}\right\|_{-1/2,\omega,\Gamma} \le C\|u\|_{1,\omega,\Omega}.$$

Remark :

a) These estimates am optimal with mspect to $|\omega|$

b) Note that estimates b) (ii) and c) (ii) yield estimates b) (i) and c) (i) in Lemma 4.8, mspec- tively, via Remark.

The following two lemmas are consequences of Lemma 8.

Lemma 9 (Mapping Properties, Classical Norms, Helmholtz Problem)

Let Ω be a Lipschitz domain. Let $p \in H^{-1/2}(\Gamma)$ and $\varphi \in H^{1/2}(\Gamma)$. Then

$$\|S_\omega[p]\|_{H^1(\Omega)} \le C|\omega|\|p\|_{H^{-1/2}(\Gamma)} \quad (31)$$

$$\|V_\omega[p]\|_{H^{1/2}(\Gamma)} \le C|\omega|\|p\|_{H^{-1/2}(\Gamma)} \quad (32)$$

$$\|K'_\omega[p]\|_{H^{-1/2}(\Gamma)} \le C|\omega|^{3/2}\|p\|_{H^{-1/2}(\Gamma)} \quad (33)$$

$$\|D_\omega[\varphi]\|_{H^1(\Omega)} \le C|\omega|^{3/2}\|\varphi\|_{H^{1/2}(\Gamma)} \quad (34)$$

$$\|K_\omega[\varphi]\|_{H^{1/2}(\Gamma)} \le C|\omega|^{3/2}\|\varphi\|_{H^{1/2}(\Gamma)} \quad (35)$$

$$\|W_\omega[\varphi]\|_{H^{-1/2}(\Gamma)} \le C|\omega|^2\|\varphi\|_{H^{1/2}(\Gamma)} \quad (36)$$

with $C = C(\Gamma, \sigma_0)$.

The Single Layer operator V_ω is further bounded independently of $|\omega|$ when it is considered as an operator mapping from $L^2(\Gamma)$ to $L^2(\Gamma)$ in three space dimensions: For $p \in L^2(\Gamma)$ there holds

$$\|V_\omega[p]\|_{L^2(\Gamma)} \le C\|p\|_{L^2(\Gamma)} \quad (37)$$

with $C = C(\Gamma)$.

In two space dimensions, one can replace the bound $C\|p\|_{L^2(\Gamma)}$ in (37) by $\tilde{C}|\omega|^{-1/3}\|p\|_{L^2(\Gamma)}$, with $\tilde{C} = \tilde{C}(\Gamma)$ again.

Note that. $|\omega|^{-1/3} \le \sigma_0^{-1/3}$, and therefore (37) also holds in two space dimensions, with $C = C(\Gamma, \sigma_0)$.

Lemma 10 (Mapping Properties, $|\omega|$-Dependent Norms, Helmholtz Problem)

Let Ω be a Lipschitz domain. Let $p \in H^{-1/2}(\Gamma)$ and $\varphi \in H^{1/2}(\Gamma)$. Then

$$\|S_\omega[p]\|_{1,\omega,\Omega} \le C|\omega|\|p\|_{-1/2,\omega,\Gamma} \quad (38)$$

$$\|V_\omega[p]\|_{1/2,\omega,\Gamma} \le C|\omega|\|p\|_{-1/2,\omega,\Gamma} \quad (39)$$

$$\|K'_\omega[p]\|_{-1/2,\omega,\Gamma} \le C|\omega|\|p\|_{-1/2,\omega,\Gamma} \quad (40)$$

$$\|D_\omega[\varphi]\|_{1,\omega,\Omega} \le C|\omega|\|\varphi\|_{1/2,\omega,\Gamma} \quad (41)$$

$$\|K_\omega[\varphi]\|_{1/2,\omega,\Gamma} \le C|\omega|\|\varphi\|_{1/2,\omega,\Gamma} \quad (42)$$

$$\|W_\omega[\varphi]\|_{-1/2,\omega,\Gamma} \le C|\omega|\|\varphi\|_{1/2,\omega,\Gamma} \quad (43)$$

with $C = C(\Gamma, \sigma_0)$.

We define bilinear forms $a_\omega^V(\cdot,\cdot)$ and $a_\omega^W(\cdot,\cdot)$ by

$$a_\omega^V(p,q) := \langle -i\omega V_\omega[p], q\rangle \quad (44)$$

for $p, q \in H^{-1/2}(\Gamma)$, and

$$a_\omega^W(\varphi,\psi) := \langle W_\omega[\varphi], -i\omega\psi\rangle \quad (45)$$

for $\varphi, \psi \in H^{1/2}(\Gamma)$. Then there hold the coercivity estimates stated in Lemma 11.

Lemma 11 (Coercivity Estimates, Helmholtz Problem Bilinear Forms)

a) Let $p \in H^{-1/2}(\Gamma)$. Then, with $C = C(\Gamma, \sigma_0)$,

(i) $\Re\left(a_\omega^V(p,p)\right) \ge C|\omega|^{-1}\|p\|^2_{H^{-1/2}(\Gamma)}$.

(ii) $\Re\left(a_\omega^V(p,p)\right) \ge C\|p\|^2_{-1/2,\omega,\Gamma}$.

b) Let $\varphi \in H^{1/2}(\Gamma)$. Then, with $C = C(\Gamma, \sigma_0)$,

(i) $\Re\left(a_\omega^W(\varphi,\varphi)\right) \ge C\|\varphi\|^2_{H^{1/2}(\Gamma)}$.

(ii) $\Re\left(a_\omega^W(\varphi,\varphi)\right) \ge C\|\varphi\|^2_{1/2,\omega,\Gamma}$.

The continuity estimates stated in Corollary are an immediate consequence of Lemmas 9 and 10.

Corollary (Continuity Estimates, Helmholtz Problem Bilinear Forms)

a) Let $p, q \in H^{-1/2}(\Gamma)$. Then, with $C = C(\Gamma, \sigma_0)$,

(i) $|a_\omega^V(p,q)| \le C|\omega|^2\|p\|_{H^{-1/2}(\Gamma)}\|q\|_{H^{-1/2}(\Gamma)}$.

(ii) $|a_\omega^V(p,q)| \le C|\omega|^2\|p\|_{-1/2,\omega,\Gamma}\|q\|_{-1/2,\omega,\Gamma}$.

b) Let $\varphi, \psi \in H^{1/2}(\Gamma)$. Then, with $C = C(\Gamma, \sigma_0)$,

(i) $|a_\omega^W(\varphi,\psi)| \le C|\omega|^3\|\varphi\|_{H^{1/2}(\Gamma)}\|\psi\|_{H^{1/2}(\Gamma)}$.

(ii) $|a_\omega^W(\varphi,\psi)| \le C|\omega|^2\|\varphi\|_{1/2,\omega,\Gamma}\|\psi\|_{1/2,\omega,\Gamma}$.

We note that we gain one power of $|\omega|$ in the coercivity estimates for $a_\omega^V(\cdot,\cdot)$, but none for the continuity estimate when $|\omega|$ -dependent Sobolev spaces are used instead of classical Sobolev spaces. Regarding $a_\omega^W(\cdot,\cdot)$, it is the other way round.

In both cases, the presence of powers of $|\omega|$ in the continuity estimates and their absence in the coercivity estimates means that we have coercivity and continuity on two different spaces in the space-time framework. In the frequency domain though, $|\omega|$is just another constant, and one can apply the Lax-Milgram Theorem as usual in this context.

The inverse operator to V_ω Ls denoted by N^ω in A. Bamberger and T. Ha-Duong (1986). As mentioned in the proof of, there hold $V_\omega N^\omega = \mathrm{Id}|_{H^{1/2}(\Gamma)}$ and $N^\omega V_\omega = \mathrm{Id}|_{H^{-1/2}(\Gamma)}$ and hence we simply write V_ω^{-1} instead of N^ω here, in particular to avoid confusion with the Newton potential, which is defined below. Some properties of V_ω^{-1} are collected in Lemma 12.

Lemma 12 (Boundedness and Coercivity of V_ω^{-1})

Let $g \in H^{1/2}(\Gamma)$. Then

$$\|V_\omega^{-1}[g]\|_{H^{-1/2}(\Gamma)} \le C|\omega|^2\|g\|_{H^{1/2}(\Gamma)} \quad (46)$$

$$\|V_\omega^{-1}[g]\|_{-1/2,\omega,\Gamma} \le C|\omega|\|g\|_{1/2,\omega,\Gamma} \quad (47)$$

$$\Re\left\langle V_\omega^{-1}[g], -i\omega g\right\rangle \ge C\|g\|^2_{H^{1/2}(\Gamma)} \quad (48)$$

$$\Re\left\langle V_\omega^{-1}[g], -i\omega g\right\rangle \ge C\|g\|^2_{1/2,\omega,\Gamma} \quad (49)$$

with $C = C(\Gamma, \sigma_0)$.

Equations (47) and (49) are proven below. A bound similar to (46) on W_ω^{-1} can be found.

Proof (of equations (47) and (49) in Lemma 12) (49) follows just as in the proof, by using Lemma 8 a) (ii) instead of (i).

To prove (47), we modify the proof, which is Lemma 8 c) (i), we take (ii). The term $\|u\|_{1,\omega,\Omega}$ can be estimated. Hence we obtain $\left\|\frac{\partial u}{\partial n}\right\|_{-1/2,\omega,\Gamma} \le C\|u\|_{1,\omega,\Omega} \le C|\omega|\|g\|_{1/2,\omega,\Gamma}$ and thus (47).

With regard to the inhomogeneous Helmholtz equation the Newton potential (or volume potential) is given

$$N_\omega[f](x) = \int_\Omega G_\omega(|x-y|)\ f(y)\ dy \tag{50}$$

for $x \in \mathbb{R}^N$. Correspondingly, the Newton potential for the wave equation is

$$N[f](x,t) = \int_{\mathbb{R}_{\ge 0}} \int_\Omega G(t-s, |x-y|)\ f(y,s)\ dy\ dt \tag{51}$$

for $(x,t) \in \mathbb{R}^N \times \mathbb{R}_{\ge 0}$. To simplify the notation, and in contrast to (P) and (HH), Ω denotes a bounded domain here. In the context of (P) and (HH), Ω would be the scatterer, which is denoted by Ω^- there.

Melenk and Sauter show that $\omega^4\|N_\omega[f]\|^2_{L^2(\Omega)} + \omega^2|N_\omega[f]|^2_{H^1(\Omega)} + |N_\omega[f]|^2_{H^2(\Omega)} \le C|\omega|^2\|f\|^2_{L^2(\Omega)}$ for $f \in L^2(\Omega)$, with $C = C(R, \sigma_0)$ respectively $C = C(\Gamma, \sigma_0)$. As an immediate consequence we obtain a bound on N_ω.

Lemma 13 (Boundedness of N_ω)

Let $f \in L^2(\Omega)$. Then

$$\|N_\omega[f]\|_{2,\omega,\Omega} \le C|\omega|\|f\|_{L^2(\Omega)} \tag{52}$$

with $C = C(\Gamma, \sigma_0)$.

Generalised Mapping Properties -

Up to now we have only presented results on the boundedness of the Helmholtz boundary layer potentials and boundary integral operators with respect to their respective natural energy spaces $H^1(\Omega)$, $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. However, generalised mapping properties are of interest as well, in particular in the context of a posteriori error estimation. The groundbreaking work on the boundary integral operators for a class of elliptic problems that includes the Laplace, Helmholtz and Lame problem was done. He subsequently extended his analysis to the boundary integral operators for the heat equation.

It is well known that the Helmholtz boundary integral operators have the same mapping properties as their Laplace counterparts. What is unknown though is how the respective estimates depend on the wave number ω when the spaces are not the respective natural energy spaces. Here we mimic Costabel's arguments in order to obtain bounds which are explicit in $|\omega|$. We begin with a generalisation of Lemma 13.

CONCLUSION

This proposition fundamentally makes two contributions to the field of time domain Boundary Element Methods. On the hypothetical side, it gives summed up mapping results to the administrators included and acquaints a posteriori mistake gauges with this field. On the implementational side, it gives a full integration scheme that can be utilized with non-uniform networks and presents an adaptable self-adaptive algorithm that permits refinements in both the spatial and worldly direction.

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