A Study on Poles and Zeros and Meromorphic Functions with Non-Zero Derivatives and Related Polynomial Differentials

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Abstract – In the study, we concentrated on the Uniqueness Theorems and Meromorphic Structure Deficiencies, Growth limitation regarding poles and zeros and meromorphic functions with non-zero subordinates and related polynomial differentials and Fix points normal and characteristic groups of some homogeneous polynomials. Additionally we study Angular conveyance of meromorphic functions concerning homogeneous and differential polynomials.

Keywords: Meromorphic, Poles, Zeros, Non-Zero Derivatives

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INTRODUCTION

Delimitation of Growth in terms of Poles and Zeroes of f $^{(k-l)}$ and f $^{(k+1)}$ and Meromorphic functions with nonzero derivatives

W.K. Hayman, (1958) has proved the following Theorems.

Theorem. Suppose that f(z) is meromorphic in $|z| \le r$ and not of the forms e^{az+b} or $(az+b)^{\lambda}$, for a complex λ. Then if $\Phi(z)=\frac{f(z)}{f'(z)}$, we have

$$T(r,\Phi) \le 3\overline{N}(r,f)+7\overline{N}(r,\frac{1}{f})+4\overline{N}(r,\frac{1}{f''})+S_{(l)}(r,\Phi)$$

Where $S_{(l)}(r,\Phi)$ is defined as in Lemma, below.

Theorem. The only functions f (z) In the plane Meromorphic, So f just has a finite number of zeros and poles formed

$$f(z)=\frac{P_1(z)}{P_2(z)}e^{P_3(z)}$$

Where $P_1(z)$, $P_2(z)$ and $P_3(z)$ are polynomials. Of these the only functions for which f and f have no zeros are e^{az+b} or $(az+b)^{-n}$, where n is a positive integer.

Theorem. Suppose the f(z) in the plane is meromorphic, that f,f ' and f have no zeros and in addition that f(z) has finite order or more generally that

$$\liminf_{r\to\infty}\frac{\log n(r,f)}{\log r}<+\infty .$$

Then,

$$f(z)=e^{az+b} \text{ or } (az+b)^{-n}.$$

We wish to prove the following interesting improvements of the above mentioned theorems.

Theorem: Suppose that $f^{(k-l)}$ is meromorphic in $|z| \le r$ and not of the forms $f(z)=\frac{1}{a^{k-l}}\frac{(az+b)^{\lambda+k-l}}{(\lambda+1)......(\lambda+k-1)}+P_1(z)$ or $f=\frac{1}{a^{k-l}}e^{az+b}+P_2(z)$. for a complex λ. (or equivalently $f^{(k-l)}$ is not of the forms $e^{(az+b)}$ or $(az+b)^{-n}$ for some complex λ).

Then if, $\Phi(z)=\frac{f^{(k-l)}(z)}{f^{k}(z)}$, $k \ge 1$,we have

$$T(r,\Phi) \le 3\overline{N}(r,f)+7\overline{N}(r,\frac{1}{f^{(k-l)}})+4\overline{N}(r,\frac{1}{f^{(k+l)}})+S_{(l)}(r,\Phi)$$

Where $S_{(l)}(r,\Phi)$ is defined as in Lemma, below.

To prove the above Theorem, we required the following lemmas.

Lemma. If 1 is a positive integer and $\Phi(z)$ is meromorphic $|z| \le r$ and is not a polynomial of degree 1 or less, then

$$T(r,\Phi) < (2+\frac{1}{l})N(r,\frac{1}{\Phi}) + (2+\frac{2}{l})\overline{N}(r,\frac{1}{\Phi^{(l)}-1}) + S_{(l)}(r,\Phi)$$

Where

$$S_{(l)}(r,\Phi) = (2+\frac{2}{l})m(r,\frac{\Phi^{(l+1)}}{\Phi^{(l)}-1}) + (2+\frac{1}{l})\left(2m(r,\frac{\Phi^{(l+1)}}{\Phi^{(l)}}) + m(r,\frac{\Phi^{(l)}}{\Phi})\right.$$

$$+\frac{1}{l}m(r,\frac{\Phi^{(l+2)}}{\Phi^{(l+1)}}) + 18 + (2+\frac{1}{l})\log\left|\frac{\Phi(0)(\Phi^{(l)}(0)-1)}{\Phi^{(l+1)}(0)}\right|$$

$$+\frac{1}{l}\log\left|\frac{\Phi^{(l+1)}(0)|\Phi^{(l)}(0)-1|}{(l+1)\Phi^{(l+2)}(0)\Phi^{(l)}(0)-1-(l+2)\Phi^{(l+1)}(0)^2}\right|,$$

If $\Phi(0) \ne 0,\infty$, $\Phi^{(l)}(0) \ne 1$, $\Phi^{(l+1)}(0) \ne 0$ and

$$(l+1)\Phi^{(l+2)}(0)(\Phi^{(l)}(0)-1) - (l+2)\Phi^{(l+1)}(0)^2 \ne 0,$$

With minor modifications otherwise.

Lemma : Where f(z) is meromorphic in $|z| < R \le \infty$ plus is not polynomial of degree 1 or less, and if $\lim\operatorname{Sup}_{r\to R} \frac{T(r,f)}{\log\frac{1}{R-r}} = +\infty$ in case R finite, then we have

$$\lim\inf_{r\to R} \frac{S_{(l)}(r,f)}{T(r,f)} = 0,$$

Where $S_{(l)}(r,f)$ is defined as in Lemma, with f(z) instead of $\Phi(z)$.

Lemma: If f(z) has meromorphic effect in $|z| < R$ And is not degree 1 or less of a polynomial, then

$$m(r,\frac{f^{(l)}}{f}) = S(r,f),$$

$$T(r,f^{(l)}) \le (l+1)T(r,f) + S(r,f)$$

And

$$m(r,\frac{f^{(l+1)}}{f^{(l)}}) + m(r,\frac{f^{(l+1)}}{f^{(l)}-1}) = S(r,f).$$

Proof of Theorem: In fact $\Phi(z)$ has a simple zero at each zero or pole of $f^{(k-1)}$ and no other zeros.

MEROMORPHIC FUNCTIONS WITH NONZERO DERIVATIVES

We saw in Theorem, that the only functions $f^{(k-1)}(z)$, meromorphic the plane, for which $f^{(k-1)}(z)$ and $f^{(k+1)}(z)$ have no zeros plus $f^{(k-1)}(z)$ Has only finite pole numbers $\frac{1}{a^{k-1}}\frac{(az+b)^{\lambda+k-1}}{(\lambda+1)......(\lambda+k-1)} + P_1(z)$ plus $\frac{1}{a^{k-1}}e^{az+b} + P_2(z)$.

Here we have been unable to prove a theorem of this type with no restriction on the poles, but if we assume also that $f^{(k-1)}(z) \ne 0$, we can weaken the condition on the poles.

Theorem: Suppose that $f^{(k-1)}(z)$ is meromorphic in the plane that $f^{(k-1)}(z)$, $f^{(k)}(z)$ and $f^{(k+1)}(z)$ have no zeros, and in addition that $f^{(k-1)}(z)$, $f^{(k)}(z)$ finite order or more generally that

$$\lim\inf_{r\to\infty} \frac{\log n(r,f)}{\log r} < +\infty.$$

Then

$$f^{(k-1)}(z) = \frac{1}{a^{k-1}}\frac{(az+b)^{\lambda+k-1}}{(\lambda+1)......(\lambda+k-1)} + P_1(z)$$

Or

$$\frac{1}{a^{k-1}}e^{az+b} + P_2(z)$$

Proof: We are assuming f(z) is transcendental, since otherwise the follows from Theorem, set $\Phi(z) = \frac{f^{(k-1)}}{f^{(k)}}$ and have by Theorem and Lemma for all r except a set of finite measure

$$T(r,\Phi) \le 3\overline{N}(r,f) + 7\overline{N}(r,\frac{1}{f^{(k-1)}}) + 4\overline{N}(r,\frac{1}{f^{(k+1)}}) + S(r,f)$$

And

$$T(r,f^{(l+1)}) \le (l+2)T(r,f) + 0(1),$$

$$T(r,\Phi') < 2T(r,\Phi) < 3T(r,\Phi) < 21\left(N(r,\frac{1}{f^{(k-1)}}) + N(r,\frac{1}{f^{(k+1)}}) + N(r,f)\right) + S(r,f)$$

Hence our hypothesis gives

$$T\left(r, \frac{f^{(k-1)}(z)f^{(k+1)}(z)}{(f^{(k)}(z))^2}\right) = T(r,\Phi') + 0(1) < Cr^s, \quad \ldots (14)$$

For a sequence $r = r_p \to \infty$. In fact it ρ_p is a sequence such that $n(\rho_p, f) < \rho_\gamma^k$ and hence $N(\rho_p, f) \le \rho_\gamma^{k-1}\log\rho_\gamma + O(1)$, then (14) holds as $r \to \infty$ through the series of interval $\frac{\rho_\gamma}{2} \le r \le \rho_\gamma$, outside a set of finite measure. Now

$$\Phi'(z) - 1 = \frac{f^{(k-1)}(z)f^{(k+1)}(z)}{(f^{(k)}(z))^2}.$$

At the poles of $f^{(k-1)}(z)$ the right hand side remains regular and different form zero. Also by hypothesis $f^{(k-1)}(z), f^{(k)}(z)$ and $f^{(k+1)}(z)$ have no zeros so that we deduce that

$$\frac{f^{(k-1)}(z)f^{(k+1)}(z)}{(f^{(k)}(z))^2} = e^{P(z)},$$

Where P (z) is an integral function. Further in view of (14), P(z) must be polynomial.

Now since $f^{(k-1)}(z) \neq 0$ we may set $f^{(k-1)}(z) = \frac{1}{g^{(k-1)}(z)}$, $\quad \ldots (15)$

Where g (z) is an integral function and a simple calculation gives

$$\frac{f^{(k-1)}(z)f^{(k+1)}(z)}{(f^{(k)}(z))^2} = 2 - \left(\frac{g^{(k-1)}(z)g^{(k+1)}(z)}{(g^{(k)}(z))^2}\right)$$

Differentiating,

$$f^{(k)}(z) = -\frac{g^{(k)}(z)}{(g^{(k-1)}(z))^2} \quad \ldots (16)$$

$$f^{(k+1)}(z) = -\frac{(g^{(k-1)}(z))^2 g^{(k+1)}(z) - g^{(k)}(z)2g^{(k-1)}(z)g^{(k)}(z)}{(g^{(k-1)}(z))^4}$$

$$f^{(k+1)}(z) = -\left(\frac{g^{(k+1)}(z)}{(g^{(k-1)}(z))^2} - 2\frac{(g^{(k)}(z))^2}{(g^{(k-1)}(z))^3}\right). \quad \ldots (17)$$

From (15), (16) and (17), we get

$$\frac{f^{(k-1)}(z)f^{(k+1)}(z)}{(f^{(k)}(z))^2} = \frac{-\left(\frac{1}{g^{(k-1)}(z)}\right)\left(\frac{g^{(k-1)}(z)g^{(k+1)}(z) - 2(g^{(k)}(z))^2}{(g^{(k-1)}(z))^3}\right)}{\frac{(g^{(k)}(z))^2}{(g^{(k-1)}(z))^4}}$$

$$= -\left(\frac{g^{(k-1)}(z)g^{(k+1)}(z) - 2(g^{(k)}(z))^2}{(g^{(k-1)}(z))^4}\right) \times \frac{(g^{(k-1)}(z))^4}{(g^{(k)}(z))^2}$$

$$= -\left(\frac{g^{(k-1)}(z)g^{(k+1)}(z) - 2(g^{(k)}(z))^2}{(g^{(k)}(z))^2}\right)$$

$$\frac{f^{(k-1)}(z)f^{(k+1)}(z)}{(f^{(k)}(z))^2} = 2 - \left(\frac{g^{(k-1)}(z)g^{(k+1)}(z)}{(g^{(k)}(z))^2}\right)$$

$$= 2 - e^{P(z)}.$$

Thus we deduce that $g^{(k-1)}(z)$ is an integral function such that

$$\frac{g^{(k-1)}(z)g^{(k+1)}(z)}{(g^{(k)}(z))^2} = 2 - e^{P(z)}$$

If P(z) is a constant, we deduce just as in Theorem, that $f^{(k-1)}(z)$ reduces to $f(z) = \frac{1}{a^{k-1}}\frac{(az+b)^{\lambda+k-1}}{(\lambda+1)\ldots\ldots(\lambda+k-1)} + P_1(z)$ or $f = \frac{1}{a^{k-1}}e^{az+b} + P_2(z)$. So we assume that P(z) has degree at least 1 and shall obtain a contradiction.

This contradiction arises from the following three Lemmas

Lemma: If P(z) is a non-constant polynomial and ω, τ, γ, then exits constants C_1, C_2, such that if $0 < \in < C_1$ the inequality

$$\left|e^{P(z)} - \omega\right| > \in |\omega|$$

Holds for large R in the annulus $R < |z| < 2R$ outside a set of circles whose radii are at most C_2 Rs.

We consider first the case when P(z) = z. For given z let z_0 be the nearest to z of the equation $e^{z_0} = \omega$, and set $z = z_0 + h$. Then

$$\left|e^z - \omega\right| = \left|e^{z_0+h} - \omega\right| = |\omega|\left|e^h - 1\right|.$$

Assume now that $|e^z-\omega|\le\frac{|\omega|}{2}$ so that $|e^h-1|\le\frac{1}{2}$, we set $h=\alpha+i\beta$ and the definition of z_0 we have $|\beta|\le\pi$. Then

$$|e^h-1|^2=(e^\alpha\cos\beta-1)^2+e^{2\alpha}\sin^2\beta=1+e^{2\alpha}-2e^\alpha\cos\beta$$

$$=(e^\alpha-1)^2+4e^\alpha\sin^2\frac{\beta}{2}.$$

Now $e^\alpha-1=\alpha e^{9\alpha}$, where $0<\theta<1$ and so if $|e^h-1|\le\frac{1}{2}$ we have $e^\alpha>\frac{1}{2}$ and so $\alpha<2|e^\alpha-1|$. Thus if

$$|e^h-1|\le\frac{1}{2},\ |e^z-\omega|\le\frac{1}{2}|\omega|,$$

We have

$$|\alpha|<2|e^\alpha-1|\le2|e^h-1|.$$

Again since

$$\left|\frac{\beta}{2}\right|\le\frac{\pi}{2}$$

We have

$$|\beta|\le\pi\left|\sin\frac{\beta}{2}\right|\le\frac{\pi}{2}e^{\frac{\alpha}{2}}|e^h-1|\le\pi|e^h-1|.$$

Thus

$$|h|=|\alpha+i\beta|\le|\alpha|+|\beta|\le(\pi+2)|e^h-1|=\frac{\pi+2}{|\omega|}|e^z-\omega|.$$

Hence if $0<\epsilon\le\frac{1}{2}$ we have

$$|e^z-\omega|\ge\epsilon|\omega|$$

Provided that $|z-z_0|\ge(\pi+2)\epsilon$ for every root z_0 of the equation $e^{z_0}=\omega$.

We now apply this result with our polynomial P(z) instead of z and deduce that if $0<\epsilon<\frac{1}{2}$

$$|e^{P(z)}-\omega|\ge\epsilon|\omega| \quad \ldots(18)$$

Provided that for every z_0 or the equation $e^{z_0}-\omega$ we have

$$|P(z)-Z_0|\ge(\pi+2)\epsilon. \quad \ldots(19)$$

To complete the proof of Lemma we need a subsidiary result

Lemma: Suppose that $p(z)=az^k+\ldots$ is a polynomial of degree $k\ge1$, Then there exist positive constants C_3,C_4, such that if $R>C_2$, t is any complex number and $0<\delta<1$, the inequality

$$|P(z)-t|>|a|\delta \quad \ldots(20)$$

Holds for $R<|z|<2R$ outside asset of k circles, which depend on t, but whose radii are less than $C_4\delta R^{1-k}$.

Let $z_1,z_2,\ldots,z_k$ be the roots of the equation P (z)=t. Then

$$P(z)-t=a(z-z_0)\ldots(z-z_k).$$

Hence if $|P(z)-t|\le|a|\delta<|a|$, we must have $|z-z_\gamma|<1$ For at least one value of γ. Now if C is sufficiently large we have

$$|P(z)|\le\frac{1}{2}|a||z|^k,\qquad |z>C|,$$

So that if R > max (4,C), (20) holds for R < |z| < 2R unless

$$|t|>\frac{1}{4}|a|R^k.$$

But for large t the roots of the equation P(z)=t are given approximately by

$$(1+0(1))az_\gamma^k=t\ .\ z_\gamma=(1+0(1))\left|\frac{t}{a}\right|^{\frac{1}{k}}\exp\left((\frac{\lambda+2\pi\gamma}{k})i\right),$$

$$\gamma=0\text{ to }k-1,$$

Where $\lambda=\arg(t/a)$. Thus for large t we have if $0\le\mu<\gamma\le k-1$,

$$|z_\mu-z_\gamma|>\left|\frac{t}{a}\right|^{1/k}\sin\frac{\pi}{k}.$$

Hence if $|P(z)-t|<|a|\delta$ so that (22) holds and $|z-z_\gamma|<1$ for some γ. C yields

$$|P(z)-t|>C'|t|^{(k-1)/k}|z-z_\gamma|,$$

Where C is a suitable constant. Thus if $|P(z)-t|<|a|\delta$ we deduce

$$|z-z_\gamma|<\frac{|a|\delta}{C'}|t|^{1/k-1}<\frac{|a|\delta}{C'}\left(\frac{|a|R^k}{4}\right)^{1/k-1} \quad C,\delta R^{1-k}$$

By (22). Since there is only k roots z, this proves Lemma

We can now complete the proof of Lemma Suppose that P(z) is polynomial satisfying the conditions of Lemma and that $R<|z|<2R$ Then if $R>C_1$ and C_1 is sufficiently large we have

$$\frac{1}{2}|a||z|^k < |P(z)| < 2|a||z|^k,$$

So that (19) holds for all complex Zn, except possibly those for which

$$|Z_0|<2|a|(2R)^k+(\pi+2)\epsilon. \qquad \ldots(23)$$

The number of roots Z_0 of the equation $e^{z_0}=\omega$ satisfying (23) is at most

$$\frac{1}{\pi}|2|a|(2R)^k+(\pi+2)\epsilon|+1<|a|(2R)^k,$$

for large R. For each such root Lemma 2.1.5 shows that (19) holds outside a set of k circles of radius at most $C_4(\pi+2)\epsilon/|a|R^{1-k}$. Thus the sum of the radius of all these circles for Z_0 satisfies (23) is at most

$$kC_4(\pi+2)\frac{\epsilon}{|a|}R^{1-k}|a|(2R)^k<C_z\epsilon R$$

As required. This completes the proof Lemma

We also need following Valiron's result

Lemma: If g (z) is a transcendental integral function and $z_r=re^{i\eta}$ is a point such that

$$|g(z_r)|=M(r,g)$$

Then we have as $r\to\infty$ Outside the finite logarithmic calculation set for any fixed Q

$$g^{(q)}(z_r)\sim g(z_r)\left(\frac{N(r)}{z_r}\right)^q,$$

Where N(r) is the central index of g(z).

This lemma now yields contradiction as follows:

For our integral function $g^{(k-1)}$, we know that

$$2-e^{P(z_r)}=\frac{g^{(k-1)}(z_r)g^{(k+1)}(z_r)}{(g^{(k)}(z_r))^2}.$$

Applying the above lemma, with q = k+1, and q = k, we get

$$g^{(k+1)}(z_r)\sim g(z_r)\left(\frac{N(r)}{z_r}\right)^{k+1},$$

$$g^{(k)}(z_r)\sim g(z_r)\left(\frac{N(r)}{z_r}\right)^{k}$$

And

$$g^{(k-1)}(z_r)\sim g(z_r)\left(\frac{N(r)}{z_r}\right)^{k-1}$$

Using the above results, we are led to

$$\frac{g^{(k-1)}(z_r)g^{(k+1)}(z_r)}{(g^{(k)}(z_r))^2}\sim\frac{g(z_r)\left(\frac{N(r)}{z_r}\right)^{k+1}g(z_r)\left(\frac{N(r)}{z_r}\right)^{k-1}}{(g(z_r))^2\left(\frac{N(r)}{z_r}\right)^{2k}}$$

$$2-e^{P(z_r)}=\frac{g^{(k-1)}(z_r)g^{(k+1)}(z_r)}{(g^{(k)}(z_r))^2}\to 1,$$

As $r\to\infty$, outside a set of finite logarithmic measure.

Hence given e > 0, the inequality

$$|e^{P(z_r)}-1|<\epsilon$$

Holds for $r>r_0$, Except the finite logarithmic measure set.

But this contradicts Lemma, which shows that for sufficiently small ϵ and all z on the circle |z| = r

$$\left|e^{P(z)}-1\right|\geq\epsilon$$

For a set E of values of r in the interval [R, 2R], such that

$$\int_E \frac{dr}{r} > \frac{1}{2R}(R-2C_2R\epsilon) > \frac{1}{4}$$

For all larger enough R.

Thus, We got the contradiction. This proves the theorems.

REFERENCES

Borel. E. (1987). Sur les-zeros des functions entries. Acta.Math.20, pp. 357-396.

Doeringer W. (1982). Exceptional Values of Differential Polynomials, Pacific J Math, 98(1), pp. 55-62.

Drasin D. (1969). Normal Families and The Nevanlinna Theory, Acta. Math 122, pp. 231-263.

Fang M. L. and Wei-Hong (2000). Unicity Theorems for Entire Functions Concerning Differential Polynomials. Indian J.Pure and Applied Math. 32(9), pp. 1343-1348.

Fred. Gross (1969). On Fix-points of Meromorphic Functions. Math. Scandinavica, 36, pp. 230-231.

Gopalakrishna. H.S. and Bhoosnurmath S. S. (1977). Exceptional values of linear combinations of the derivatives of a meromorphic function, Casopis pro Pestovani Mathematiky, roc. 102 praha pp. 25-29.

Hayman W.K. (1964). Meromorphic Functions, Oxford University Press, London.

Hayman W.K. (1959). Picard Values of Meromorphic Functions and Their Derivatives, Ann. of .Math, 70, pp. 9-42.

Julia G. (1919-20). Sur quelques prooperties nouveles des fonctions entires ou Meromorphes. Ann. Ecole. Norm Sup.36, pp. 93-126, 37, 165-218.

Ku Y. (1978). Sur les families Normales de Fonctions meromorphes. Sci.Sinica, 21, pp. 431-445.

Langley J. K. (1984). Normal Families and A Result on Drasin Theory, Proc. Roy. Soc. Edin. 98 A. pp. 325-395.

Nevanlinna R. (1993). Analytic Functions, Springer Veriag, 1970. 13.Schiff Joe L, Normal Families, Springer Verlag, New York.

Schwick W. (1989). Criteria for Families of Meromorphic Functions. J. D'Analyse. Math 52, pp. 241-289.

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