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Some Results on Pell's Equation

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Abstract – In this paper we shall define and discuss about some special characters of Pell's equation and a very simple verified results on Pell's equation to show that, one can find more solution of Pell's equation which give rise to next some another solution by continued fraction in a very simple manner. With the help of convergence of continued fraction expansion $\sqrt{d}$ of in the Pell's equation $x^2-dy^2=1$, we shall find out Diophantine pair solution of Pell's equation under certain condition. At last we shall get some recent results on Pell's equation.

Keywords – Congruent Numbers, Continued Fraction, Convergent, Pell's Equation.

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INTRODUCTION

An English mathematician John Pell (1611-1685), who taught mathematics in Holland, at the universities of Amsterdam and Breda in 1640's had introducing a newly type Diophantine equation which latterly famous as Pell's equation. It has a long attractive history, [4].

One of the main reasons for the popularity of Pell's equation is the fact that many natural questions that one might ask about integer's leads to a quadratic equation in two variables, which can be casted as a Pell's equation,[1].

Fermat was also paying attention in the Pell's equation and has given ideas on some of the basic theories regarding Pell's equation. It was Lagrange, who discovered the complete theory of the equation $x^2-dy^2=1$, [5].

Euler erroneously named the equation to John Pell. He did so apparently because Pell was influential in writing a book containing these equations. World fame Indian astronomer and mathematician Brahmagupta has left us with this intriguing challenge, "A person who can, within a year, solve $x^2-92y^2=1$ is a mathematician."[2].

In general Pell's equation is a Diophantine equation of the form $x^2-dy^2=1$, where is a positive non square integer and has a long fascinating history and its applications are wide and Pell's equation always has the trivial solution (x, y) = (1,0), and has infinite solutions and many problems can be solved using Pell's equation,[3].

RESULTS ON PELL'S EQUATION

The Pell equation is a Diophantine equation of the form $x^2-dy^2=1$. We want to find all integer pairs (x,y) that satisfy the equation. Since any given d solution (x,y) yields multiple solution we restrict ourselves to those solutions $(\pm x,\pm y)$ where x and y are positive integers. We generally take to be a positive non-square integer; otherwise there are only uninteresting solutions.

If $d<0$ then $(x,y)=(\pm 1,0)$ in the case $d<-1$ and $(x,y)=(0,\pm 1)$ or $(\pm 1,0)$ in the case $d=-1$, if $d=0$, then $x=\pm 1$ (y arbitrary) and if d is non-zero square, then dy and x are consecutive squares, implying that $(x,y)=(\pm 1,0)$. One of the main property of the Pell equation has always the trivial solution $(x,y)=(1,0)$.

The following result is well known

If $\frac{p_n}{q_n}$ is the n^{th} convergent to the irrational number x, then

$$\left|x-\frac{p_n}{q_n}\right|<\frac{1}{q_{n+1}q_n}\leq\frac{1}{q_n{}^2}.$$

Theorem 1.1. If $\sqrt{d}$ is a convergent of the continued fraction expansion of $\frac{p}{q}$ then (p,q) is a one of the solution of equation $x^2 - dy^2 = 1$. Where $1 + 2q\sqrt{d} > 1$

Proof. If $\frac{p}{q}$ is a convergent of $\sqrt{d}$, then

$$\left|\sqrt{d} - \frac{p}{q}\right| < \frac{1}{q^2}$$

And, therefore

$$\left|p - q\sqrt{d}\right| < \frac{1}{q}.$$

Now

$$\left|p + q\sqrt{d}\right| = \left|p - q\sqrt{d} + 2q\sqrt{d}\right| \le \left|p - q\sqrt{d}\right| + \left|2q\sqrt{d}\right|$$

$$< \frac{1}{q} + \left|2q\sqrt{d}\right| \le \left(1 + 2q\sqrt{d}\right)q.$$

These two inequalities combine to yield

$$|p^2 - dq^2| = |p - q\sqrt{d}| \times |p + q\sqrt{d}| < \tfrac{1}{q}(1 + 2q\sqrt{d})q = 1 + 2q\sqrt{d}.$$

Example 1.1. Let $d = 7$. Then

$$\sqrt{7} = [2, \overline{1,1,1,4}].$$

The first two convergent of $\sqrt{7}$ are $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \ldots$

Now calculating, $p_n^2 - 7q_n^2$, we find that

$$2^2 - 7\cdot1^2 = -3, 3^2 - 7\cdot1^2 = 2, 5^2 - 7\cdot2^2 = -3, 8^2 - 7\cdot3^2 = 1.$$

Hence $x = 8, y = 3$ provides a positive solution of $x^2 - 7y^2 = 1$

Corollary: If d is a positive integer and it is not a perfect square, then the continued fraction expansion of $\sqrt{d}$ necessarily has the from

$$\sqrt{d} = [a_0, \overline{a_1, a_2, a_3, \ldots, a_3, a_2, a_1, 2a_0}]$$

Theorem 1.2. Let (x_1, y_1) be the fundamental solution of $x^4 - dy^4 = 1$. And every pair of integers (x_n, y_n) defined by the following condition.

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n,$$

And

$$x_n - y_n\sqrt{d} = (x_1 - y_1\sqrt{d})^n$$

Then this (x_n, y_n) will be also a positive solution, where n=1,2,3,….

Proof. Further, because x_1 and y_1 are positive x_n and y_n are both positive integers. Since (x_1, y_1), is a solution of $x^2 - dy^2 = 1$, we have

$$\begin{aligned} x_n^2 - dy_n^2 &= (x_n + y_n\sqrt{d})(x_n - y_n\sqrt{d}) \\ &= (x_1 + y_1\sqrt{d})^n (x_1 - y_1\sqrt{d})^n \\ &= (x_1^2 - dy_1^2)^n \\ &= 1^n \\ \therefore x_n^2 - dy_n^2 &= 1. \end{aligned}$$

Hence, (x_n, y_n) is a solution.

Theorem 1.3. If and $x_1 > 1, y_1 \ge 1$ and $x_n - y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$, then $x_{n+1} > x_n$ and $y_{n+1} > y_n$ for positive n.

Proof. We will prove this by induction, observe that $x_2 = x_1^2 + dy_1^2$ and $y_2 = 2x_1y_1$ Since $x_1 > 1, y_1 > 1$, and d is a positive integer it is clear that $x_2 > x_1, y_2 > y_1$. So, the result holds for $n = 1$.

Now assume the solution (x_n, y_n) with x_n and y_n positive integers greater than 1. We have,

$$\begin{aligned} x_{n+1} + y_{n+1}d &= (x_1 + y_1\sqrt{d})^{n+1} \\ &= (x_1 + y_1\sqrt{d})(x_1 + y_1\sqrt{d})^n \\ &= (x_1x_n + dy_1y_n) + (x_1y_n + x_ny_1)\sqrt{d} \end{aligned}$$

Therefore, $x_{n+1} = x_1x_n + dy_1y_n$ and $y_{n+1} = x_1y_n + x_ny_1$. We know that, $x_1x_n > x_n$ and $dy_1y_n > 0$, so $x_{n+1} - x_1x_n - dy_1y_n > x_n$ also $x_1y_n > y_n$ and $x_ny_1 > 0$ implies that $y_{n+1} = x_1y_n - x_ny_1 > y_n$. Therefore, we have $x_{n+1} > x_n$ and $y_{n+1} > y_n$.

CONCLUSION

Hence we have seen simple method for constructing some results and relation between trivial solutions of the Pell's equation under some certain conditions with its corresponding complement of solution which give rise to new idea about solution pairs.

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