A Research on Common Fixed Point Theorems with Generalized Spaces

Seema Devi[1]* Dr. Ashwani Kumar[2]

[1]Research Scholar, OPJS University, Churu, Rajasthan

[2]Associate Professor, OPJS University, Churu, Rajasthan

Abstract – The present investigation is fundamentally worried about a few new sorts of fixed point theorems in various spaces, for example, cone metric spaces and fuzzy metric spaces. By utilizing these acquired fixed point theorems, we at that point demonstrate the presence and uniqueness of the solutions to two classes of two-point customary differential equation problems.

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INTRODUCTION

After the observed Banach contraction principle (BCP) in 1922, there have been various results in the writing managing mappings fulfilling the contraction conditions of different kinds including even nonlinear articulations. One of its vital generalizations is given by Jungck (1976) in which he built up regular fixed point theorems for driving pair of maps. This idea of commutativity of maps is casual to weakly commutativity, similarity, weakly similarity, and so forth.. Branciari (2002) acquired a fixed point hypothesis for map fulfilling a simple of Banach contraction principle for integral sort imbalance. This result was additionally generalized by numerous creators.

The motivation behind this section is to enhance and generalize some current normal fixed point results in generalized metric spaces and $\theta-\mathcal{L}$- fuzzy metric spaces for the maps fulfilling integral write contractive conditions. To begin with, we acquire some fixed point results for half and half match of single and multivalued maps in the settings of b - metric spaces. From there on, we characterize $\theta-\mathcal{L}$- fuzzy metric space and acquire some fixed point theorems in it. Some current results are determined as special cases.

We initially demonstrate some basic fixed point and occurrence point theorems for half breed combine of maps in b - metric spaces fulfilling basic property (E. An.) and an integral disparity.

COMMON FIXED POINT THEOREM USING INTEGRALINEQUALITY

Theorem 2.1. Let (X, d) be a complete *b* - metric space and $f, g: X \to X$ and $F, G: X \to CB(X)$ such that

(ii) $FX \subset gX,\ GX \subset fX$,

(iii) The pairs (F, f) and (G, g) satisfy the common property $(E.A)$,

(iv) for all $x, y \in X$,

$$\int_0^{h(Fx,\ Gy)} \phi(t)\, dt \le q\left(\int_0^{M(x,\ y)} \phi(t)\, dt\right) \quad (2.1)$$

where $\phi: \Re^+ \to \Re^+$ is Lebesgue integrable mapping which is summable. non-negative and such that

$$\int_0^{\varepsilon} \phi(t)\, dt > 0, \text{ for each } \varepsilon > 0 \quad (2.2)$$

and

$$M(x, y) = \max\{d(fx, gy), d(Fx, fx), d(Gy, gy), [d(Fx, gy) + d(Gy,$$
(2.3)

with $qb < 1,\ \lambda b < 1$, where $\lambda = \max\{q, \frac{qb}{2-qb}\}$. If fX and gX are closed subspace of X, then

(1) f and F have a coincidence point.

(2) g and G have a coincidence point,

(3) f and F have a common fixed point provided that f is F – weakly commuting at u and $ffu = fu$ for $u \in C(f, F)$, where, $C(f, F)$ = {.r : ,r is a coincidence point of f and F}.

(4) g and G have a ordinary fixed point provide that g is G- weakly commute at v and $ggv = gv$ for $v \in C(g, G)$.

(5) $f, g,$ F and G have a ordinary fixed point provided (3) and (4) are true.

Proof. Let $x_0 \in X$. From (i) we can construct a series $\{y_n\}$ in Xsuch that $y_{2n+1} = fx_{2n+1} \in Gx_{2n}$ $y_{2n+2} = gx_{2n+2} \in Fx_{2n+1}$ for all $n \geq 0$.

It follows from equation (2.1) that

$$\int_0^{h(y_{2n+2},\, y_{2n+3})} \phi(t)\,dt = \int_0^{h(G_{2n+1},\, F_{2n+2})} \phi(t)\,dt \leq q\left(\int_0^{M(x_{2n+1},\, x_{2n+2})} \phi(t)\,dt\right),$$

where,

$$\int_0^{h(y_{2n+2},\, y_{2n+3})} \phi(t)\,dt \leq \left(\int_0^{\max\{d(y_{2n+1},\, y_{2n+2}),\, d(y_{2n+2},\, y_{2n+3}),\, d(y_{2n+1},\, y_{2n+3})/2} \phi(t)\,dt\right)$$

$$\leq \lambda\left(\int_0^{h(y_{2n+1},\, y_{2n+2})} \phi(t)\,dt\right).$$

Thus,

$$M(x_{2n+1}, x_{2n+2}) = \max\{d(fx_{2n+1}, gx_{2n+2}), d(Fx_{2n+1}, fx_{2n+1}), d(Gx_{2n+2}, gx_{2n+2}),$$
$$[d(Fx_{2n+1}, gx_{2n+2}) + d(Gx_{2n+2}, fx_{2n+1})]/2\}$$
$$= \max\{d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+3})/2\}$$

Similarly,

$$\int_0^{h(y_{2n+1},\, y_{2n+2})} \phi(t)\,dt \leq \lambda\left(\int_0^{h(y_{2n},\, y_{2n+1})} \phi(t)\,dt\right),$$

where, $\lambda = \max\{q, \frac{qb}{2-qb}\}.$

Thus, we have proved that for all $n \geq 0$

$$\int_0^{h(y_{n+1},\, y_{n+2})} \phi(t)\,dt \leq \lambda\left(\int_0^{h(y_n,\, y_{n+1})} \phi(t)\,dt\right) \leq \lambda^n\left(\int_0^{h(y_0,\, y_1)} \phi(t)\,dt\right).$$

Hence for all and noting, a constant, we have

$m \geq n \geq 0$ C- $\lambda = \max\{q, \frac{qb}{2-qb}\}$ 7

$$\int_0^{h(y_m,\, y_n)} \phi(t)\,dt \leq \sum_{i=n}^{m-1} \int_0^{h(y_i,\, y_{i+1})} \phi(t)\,dt \leq \sum_{i=n}^{m-1} \lambda^i\left(\int_0^{h(y_0,\, y_1)} \phi(t)\,dt\right) \leq \frac{\lambda^n}{1-\lambda}\left(\int_0^{h(y_0,\, y_1)} \phi(t)\,dt\right)$$

Then

$\lim_{m,\, n\to\infty} \int_0^{h(y_m,\, y_n)} \phi(t)\,dt = 0,$ i.e., $\{y_n\}$ is a Cauchy sequence.

Since $\{y_n\}$ is a Cauchy sequence, there exist a z satisfying

$$\lim_{n\to\infty} y_n = z = \lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} gx_{2n+2}.$$

Since fX and gX are closed, there exist u, v such that $fu = z = gv.$ A similar argument proves that

$$\lim_{n\to\infty} Fx_{2n+1} = \lim_{n\to\infty} Gx_{2n+2}$$

$$z \in \lim_{n\to\infty} Fx_{2n+1} = \lim_{n\to\infty} Gx_{2n+2}.$$

If $\lim_{n\to\infty} Fx_{2n+1} = A$ and $\lim_{n\to\infty} Gx_{2n+2} = B,$ then $z \in A \cap B.$ Thus (F, f) and (g, G) satisfy, common property $(E.\ A)$

We claim that $gv \in Gv.$ To prove it, we take $y = v$ in (2.1),

$$\int_0^{h(Fx_n,\, Gv)} \phi(t)\,dt \leq q\left(\int_0^{M(x_n,\, v)} \phi(t)\,dt\right),$$

where,

$$M(x_n, v) = \max\{d(fx_n, gv), d(Fx_n, fx_n), d(Gv, gv),$$
$$[d(Fx_n, gv) + d(Gv, fx_n)]/2\}$$

Taking the limit as $n \to \infty$, we obtain

$$\int_0^{h(A,\, Gv)} \phi(t)\,dt \leq q\left(\int_0^{M(A,\, v)} \phi(t)\,dt\right),$$

where,
$M(A, v) = \max\{d(fu, gv), d(A, fu), d(Gv, gv), [d(A, gv) + d(Gv, fu)]/2\} = d(Gv, gv).$

Since $gv = fu \in A,$ it follows from the definitions of Hausdorff metric that

$$d(Gv, gv) \leq h(A, Gv) \leq d(Gv, gv),$$

which implies that $gv \in Gv.$

On the other hand, by condition (iii) again we have,

$$\int_0^{h(Fu,\,Gy_n)} \phi(t)\,dt \le q\left(\int_0^{M(u,\,y_n)} \phi(t)\,dt\right),$$

where,

$$M(u,\,y_n) = \max\{d(fu,\,gy_n),\,d(Fu,\,fy_n),\,d(Gy_n,\,gy_n),\\ [d(Fu,\,gy_n)+d(Gy_n,\,fu)]/2\}$$

and similarly, we obtain

$d(fu,\,Fu) \le h(Fu,\,B) \le d(fu,\,Fu)$. Hence $fu \in Fu.$

Thus f and *F* have a coincidence point u and g and *G* have a coincidence point v. This completes the proofs of part (1) and (2).

Furthermore, by virtue of condition (3), we obtain $ffu = fu$ and $ffu \in Ffu$. Thus $u = fu \in Fu$. This proves (3). A similar argument proves (4). Then (5) holds immediately.

If we put $\phi(t) = 1$ and $b = 1$ in Theorem 2.1. the following result of Liu et. al. (2005) is obtained.

Corollary 2.1. Let $(X,\,d)$ be a complete metric space and $f,\,g: X \to X$ and $F,\,G: X \to CB(X)$ such that

(i) $FX \subset gX,\ GX \subset fX$,

(ii) The pairs $(F,\,f)$ and $(G,\,g)$ satisfy the common property $(E.A)$.

Let $q \in (0,\,1)$ be a constant, such that for all $x \ne y$ in X,
$h(Fx,\,Gy) \le q\max\{d(fx,\,gy),\,d(Fx,\,fx),\,d(Gy,\,gy),\,[d(Fx,\,gy)+d(Gy,\,fx)]/2\}$

If *fX* and *gX* are closed subspace of X, then

(1) *f* and *F* have a coincidence point.

(2) *g* and *G* have a coincidence point,

(3) f and *F* have a common fixed point provided that f is *F* weakly commuting at u and $ffu = fu$ for $u \in C(f,\,F)$,

(4) g and *G* have a common fixed point provided that g is *G*- weakly commuting at v and $ggv = gv$ for $v \in C(g,G)$,

(5) $f,\,g,\,F$ and *G* have a common fixed point provided (3) and (4) are true.

As an application of Theorem 2.1 we obtain the following result.

Theorem 2.2. Let $(X,\,d)$ be a complete *b*- metric space and $f,\,g: X \to X$ and $F,\,G: X \to CB(X)$ such that

(i) $FX \subset gX,\ GX \subset fX$,

(ii) The pairs $(F,\,f)$ and $(G,\,g)$ satisfy the common property $(E.A.)$,

(iii) For all $x,\,y \in X$,

$$\int_0^{h(Fx,\,Gy)} \phi(t)\,dt \le q\left(\int_0^{M(x,\,y)} \phi(t)\,dt\right),$$

where $\phi: \mathfrak{R}^+ \to \mathfrak{R}^+$ is a Lebesgue integrable mapping which is suimnable, non-negative and such that $\int_0^{\varepsilon} \phi(t)\,dt > 0,$ for each $\varepsilon > 0.$

$$M(x,\,y) = \alpha\, d(fx,\,gy) + \beta\max\{d(Fx,\,fx),\,d(Gy,\,gy)\}\\ +\gamma\max\{d(Fx,\,gy)+d(Gy,\,fx),\,d(Fx,\,fx)+d(Gy,\,gy)\} \quad (2.4)$$

with $\alpha+\beta+2\gamma<1,\ qb<1,\ \lambda b<1,$ where $\lambda = \max\{q,\,\frac{qb}{2-qb}\}.$

If fX and gX are closed subspace of X, then

(1) f and F have a coincidence point,

(2) g and G have a coincidence point,

(3) f and *F* have a common fixed point provided that f is *F* - weakly commuting at u and $ffu = fu$ for $u \in C(f,\,F)$,

(4) g and *G* have a common fixed point provided that g is *G*- weakly commuting at v and $ggv = gv$ for $v \in C(g,G)$,

(5) $f,\,g,\,F$ and *G* have a common fixed point provided (3) and (4) are true.

Proof. From equation (2.4). we have

$$h(Fx,\,Gy) \le \alpha\, d(fx,\,gy) + \beta\max\{d(Fx,\,fx),\,d(Gy,\,gy)\}\\ +2\gamma\max\{[d(Fx,\,gy)+d(Gy,\,fx)]/2,[d(Fx,\,fx)+d(Gy,\,gy)]/2\}.$$

But,

$$\max\{d(Fx,\,fx),\,d(Gy,\,gy)\} \ge (d(Fx,\,fx)+d(Gy,\,gy))/2,$$

so,

$$h(Fx, Gy) \le \alpha\, d(fx, gy) + \beta \max\{d(Fx, fx), d(Gy, gy)\} + 2\gamma \max\{[d(Fx, gy) + d(Gy, fx)]/2, d(Fx, fx), d(Gy, gy)\}.$$

Let $q = \alpha + \beta + 2\gamma < 1.$ Following (2.4), it is easy to see that

$$h(Fx, Gy) \le q \max\{d(fx, gy), d(Fx, fx), d(Gy, gy), [d(Fx, gy) + d(Gy, fx)]/2\}.$$

Thus by Theorem 2.1, we arrive at the conclusions of Theorem 2.2.

If we put $\phi(t) = 1$ and $b = 1$ in Theorem 2.2, we get the following result of Liu et al(2005),

Corollary 2.2. Let (X, d) be a complete metric space and $f, g : X \to X$ and $F, G : X \to CB(X)$ such that

(i) $FX \subset gX,\ GX \subset fX$,

(ii) The pairs (F, f) and (G, g) satisfy the common property $(E.A)$.

Let $\lambda \in (0, 1)$ be a constant, such that for all $x \ne y$ in X,

$$h(Fx, Gy) \le \alpha\, d(fx, gy) + \beta \max\{d(Fx, fx), d(Gy, gy)\} + \gamma \max\{d(Fx, gy) + d(Gy, fx), d(Fx, fx) + d(Gy, gy)\}.$$

and $\alpha + \beta + 2\gamma < 1.$ If fX and gX are closed subspace of X, then

(i) f and *F* have a coincidence point.

(ii) g and *G* have a coincidence point.

(iii) f and *F* have a common fixed point provided that f is *F* - weakly commuting at u and $ffu = fu$ for $u \in C(f, F)$,

(iv) g and *G* have a common fixed point provided that g is *G*- weakly commuting at v and $ggv = gv$ for $v \in C(g, G)$,

(v) f, g, F and *G* have a common fixed point provided (3) and (4) are true.

Now we give a common fixed point result for four self mappings in a *b*- metric space.

Theorem 2.3. Let (X, d) be a complete 6-metric space and $A, B, S, T : X \to X$ be such that

(i) $AX \subset TX,\ BX \subset SX$,

(ii) The pairs (A, S) and (B, T) are weakly compatible,

(iii) for all $x, y \in X$,

$$\int_0^{d(Ax, By)} \phi(t)\, dt \le q\left(\int_0^{M(x, y)} \phi(t)\, dt\right)$$

where $\phi : \Re^+ \to \Re^+$ is a Lebesgue integrable mapping which is suimnable, non-negative and such that

$$\int_0^{\varepsilon} \phi(t)\, dt > 0, \text{ for each } \varepsilon > 0,$$

and

$$M(x, y) = \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), [d(Sx, By) + d(Ty, Ax)]/2\} \tag{2.5}$$

with $qb < 1,\ \lambda b < 1$, where $\lambda = \max\{q, \frac{qb}{2 - qb}\}.$

If TX or SX is closed, then *A. B*, 5 and *T* have a common fixed point.

Proof. Let $x_0 \in X.$ From (i) we can construct a sequence $\{y_n\}$ in *X* such that $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n+2} = Sx_{2n+2} = Bx_{2n+1}$

As in Theorem 2.1. we can prove that $\{y_n\}$ is a Cauchy sequence. Since *X* is complete, the sequence $\{y_n\}$ converges to a point z in *X*. Consequently, the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n+2}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the same limit z.

Now suppose that $T(X)$ is closed. Then since $\{Tx_{2n+1}\} \subset T(X)$, there exists a point $u \in X$ such that $z = Tu.$ Then by using (2.5), with $x = x_{2n}$ and $y = u$ we get

$$\int_0^{d(Ax_{2n}, Bu)} \phi(t)\, dt \le q\left(\int_0^{M(x_{2n}, u)} \phi(t)\, dt\right),$$

where,

$$M(x_{2n}, u) = \max\{d(Sx_{2n}, Tu), d(Sx_{2n}, Ax_{2n}), d(Tu, Bu), [d(Sx_{2n}, Bu) + d(Tu, Ax_{2n})]/2\}$$

Letting $n \to \infty$, we get

$$\int_0^{d(z, Bu)} \phi(t)\, dt \le q\left(\int_0^{M(z, u)} \phi(t)\, dt\right),$$

where,

$$M(z, u) = \max\{d(z, z), d(z, z), d(z, Bu), [d(z, Bu) + d(z, z)]/2\}.$$

Thus,

$\int_0^{d(z, Bu)} \phi(t)\,dt \le q\left(\int_0^{M(z, Bu)} \phi(t)\,dt\right),$ which is a contradiction. This implies that $z = Bu.$ Therefore $Tu = z = Bu.$ Hence by the weak compatibility of the pair (B, T) it in unediately follows that $BTu = TBu$, that is, $Bz = Tz.$ Next, we shall show that z is a common fixed point of *B* and *T*. By setting $x = x_{2n}$ and $y = z$ hi (2.5) we have

$$\int_0^{d(Ax_{2n}, Bz)} \phi(t)\,dt \le q\left(\int_0^{M(x_{2n}, z)} \phi(t)\,dt\right),$$

where,

$$M(x_{2n}, z) = \max\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz), [d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})]/2\}$$

Letting $n \to \infty,$ and noting that $\lim_{n\to\infty} Ax_{2n} = z = \lim_{n\to\infty} Sx_{2n}$ and $Bz = Tz,$ we get

$$M(z, z) = \max\{d(z, Bz), d(z\ z), d(Bz, Bz), [d(z, Bz) + d(Bz, z)]/2\} = d(z, Bz)$$

$$\int_0^{d(z, Bz)} \phi(t)\,dt \le q\left(\int_0^{M(z, z)} \phi(t)\,dt\right),$$

where,

Thus, $\int_0^{d(z, Bz)} \phi(t)\,dt \le q\left(\int_0^{d(z, Bz)} \phi(t)\,dt\right)$ which is a contradiction, so $z = Bz.$ Thus we have $z = Bz$ = $Tz,$ i.e., z is a conunon fixed point of *B* and *T*.

Further, $z = Bz$ implies that $z \in BX \subset SX.$ Therefore there exists a point $v \in X$ such that $z = Sv.$We now show that $Av = Sv.$ Indeed, by setting $x = v$ and $y = x_{2n-1}$ in (2.5) and taking $n \to \infty,$ we get $Av = z.$ Hence $Av = z = Sv.$ Then by the weak compatibility of the pair (A, S) we immediately have $SAv = Sz = ASv = Az.$ Hence $Az = Sz.$

Now. by setting $x = z$ and $y = x_{2n-1}$ in (2.5) and following the earlier arguments, it can easily be verified that z is a common fixed point of *A* and *S* as well. Hence z is a common fixed pomt of *A.B.S* and *T*.

The uniqueness of z as a common fixed point of *A, B. S* and *T* can easily be verified. In fact, if $z \ne z'$ is another common fixed point of the given mappings, then by setting $x = z$ and $y = z'$ in (2.5) we get

$$\int_0^{d(z, z')} \phi(t)\,dt = \int_0^{d(Az, Bz')} \phi(t)\,dt \le q\left(\int_0^{M(z, z')} \phi(t)\,dt\right),$$

where,

$$M(z, z') = \max\{d(Sz, Tz'), d(Sz, Az), d(Tz', Bz'), [d(Sz, Bz') + d(Tz', Az)]/2\}$$
$$\max\{d(z, z'), d(z, z), d(z', z'), [d(z, z') + d(z', z)]/2\} = d(z, z')$$

Thus we get,

$\int_0^{d(z, z')} \phi(t)\,dt \le q \int_0^{d(z, z')} \phi(t)\,dt$, a contradiction. Thus of *A, B. S* and *T*.

z=z' and z is a unique common fixed point

COMMON FIXED POINT THEOREM IN *e-£*-FUZZY METRIC SPACE-

In 1965, Zacleh (1965) presented fuzzy set, which is additionally generalized to intuitionistic fuzzy set by Atauassov (1986) and $\mathcal{L}$- fuzzy set by Goguen (1967), Thereafter a few creators characterized fuzzy metric spaces in various ways. Subsequently a few fuzzy fixed point theorems are likewise settled in their new settings.

We first review the preliminaries required for ensuing results.

Definition 2.1. Let $\theta : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a continuous map with respect to each variable. Then θ is called a 5-action if and only if it satisfies the following conditions

(i) $\theta(0, 0) = 0$ and $\theta(t, s) = \theta(s, t)$ for all $t, s \ge 0,$

(ii) $\theta(s, t) < \theta(u, v)$ if $s < u$ and $t \le v$ or $s \le u$ and $t < v,$

(iii) for each $r \in \mathrm{Im}(\theta) - \{0\}$ and for each $s \in (0, r]$, there exists $t \in (0, r]$ such that $\theta(t, s) = r,$ where $\mathrm{Im}(\theta) = \{\theta(s, t) : s \ge 0, t \ge 0\},$

(iv) $\theta(s, 0) \le s,$ for all $s > 0.$

Definition 2.2 . Let *X* be a nonempty set. A mapping $d_\theta : X \times X \to [0, \infty)$ is called a θ- metric on A' with respect to *B* - action, if d_θ satisfies the following

(i) $d_\theta(x, y) = 0$ if $x = y,$

(ii) $d_\theta(x, y) = d_\theta(y, x)$, for all $x, y \in X$,

(iii) $d_\theta(x, y) \le \theta(d_\theta(x, z), d_\theta(z, y))$, for all $x, y, z \in X$.

Then (X, d_θ) is called a θ - metric space.

Remark 2.1. If $\theta(s, t) = s + t$, then the θ - metric space becomes a metric space.

Remark 2.2. If $\theta(s, t) = b(s+t), b \ge 1$, then the θ- metric space is the 6-metric space.

Now we define $\theta - \mathcal{L}$- fuzzy metric space following the definition given by George and Veeramani,

Definition 2.3. The 3-triplet $(X, M, \mathcal{T})$ is said to be an $\theta - \mathcal{L}$-fuzzy metric space, if A' is an arbitrary (non-empty) set. $\mathcal{T}$ is a continuous t - norm 011 $\mathcal{L}$ and M is an $\mathcal{L}$ -fuzzy set 011 $X^2 \times (0, \infty)$ with respect to B~ action. $\theta \in M$ satisfying the following conditions for eveiy x, y, z in X and t, s in $(0, \infty)$

(i) $M(x, y, t) >_L 0_{\mathcal{L}}$,

(ii) $M(x, y, t) = 1_{\mathcal{L}}$ *for all* $t > 0$, iff $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$

(iv) $\mathcal{T}(M(x, y, t), M(y, z, s)) \le_L M(x, z, \theta(t, s))$,

(v) $M(x, y, .): (0, \infty) \to L$ is continuous and $\lim_{t\to\infty} M(x, y, t) = 1_{\mathcal{L}}$.

In this case M is called a $\theta - \mathcal{L}$ -fuzzy metric space. If $M = M_{M,N}$ is an intuitionistic fuzzy set, then the 3-tuple $(X, M_{M,N}, \mathcal{T})$ is said to be a θ- intuitionistic fuzzy metric space.

Example 2.1. Let $X = [-1, 2]$ endowed with the usual metric

$d(x, y) = |x - y|, \ \forall x, y \in X$ and $\theta(t, s) = t + s$. Let $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$, where $(L^*, \le_L)$ is a complete lattice and let $M_{M,N}$ be the intuitionistic fuzzv set on $X^2 \times (0, \infty)$ defined as follows

$$M_{M,N}(x, y, t) = \left(\frac{t}{t + d(x, y)}, \frac{d(x, y)}{t + d(x, y)} \right)$$

Then $(X, M_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Definition 2.4. A fuzzy metric space $(X, M, \mathcal{T})$ is said to have the property (C), if it satisfies the following condition: $M(x, y, t) = C$ for all $t > 0$ implies $C = 1$.

Lemma 2.1. Let $\phi(t): [0, \infty) \to [0, \infty)$ satisfies the following condition (ϕ) It is nondecreasing and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, where $\phi^n(t)$ denotes the n^{th} iterate of $\phi(t)$, then $\phi(t) < t$ for all $t > 0$.

Lemma 2.2. Let $(X, M, \mathcal{T})$ be a $\theta - \mathcal{L}$- fuzzy metric space. Then $M(x, y, t)$ is no decreasing with respect to t, for all x, y in X.

Proof. Suppose $M(x, y, t) >_L M(x, y, s)$ for some $0 < t < s$. Then

$$\mathcal{T}(M(x, y, t), M(y, y, s-t)) \le_L M(x, y, \theta(s-t, t)).$$

Now from the definition, we have, for each $r \in \mathrm{Im}(\theta) - \{0\}$ and for each $s \in (0, r]$, there exists $t \in (0, r]$ such that $\theta(t, s) = r$, where $\mathrm{Im}(\theta) = \{\theta(s, t) : s \ge 0, t \ge 0\}$.

So, $\theta(t, s-t) = r$, since $s - t > 0$.

Then.
$\mathcal{T}(M(x, y, t), M(y, y, s-t)) \le M(x, y, r) > M(x, y, t)$,

since $t \in (0, r]$.

Also, we have $M(y, y, s-t) = 1_{\mathcal{L}}$ for all $s > 0$.

Thus we have $M(x, y, t) > M(x, y, t)$, a contradiction.

Lemma 2.3. Let $(X, M, \mathcal{T})$ be a $\theta - \mathcal{L}$- fuzzy metric space. Define $E_{\lambda,M}: X^2 \to R^+ \cup \{0\}$ by $E_{\lambda,M}(x, y) = \inf\{t > 0 : M(x, y, t) >_L N(\lambda)\}$ for each $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $x, y \in X$. Then we have

(i) For any $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $E_{\mu,M}(x_1, x_n) \le E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \dots + E_{\lambda,M}(x_{n-1}, x_n)$ for any $x_1, \dots, x_n \in X$.

(ii) The sequence $\{x_n\}_{n \in N}$ is convergent to, r with respect to $\theta - \mathcal{L}$- fuzzy metric *M* if and only if it is Cauchy with $E_{\lambda,M}$.

Proof. For the first part, we can find $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ for every $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{T}^{n-1}(N(\lambda), \dots, N(\lambda)) \ge_L N(\mu)$.

Using the definition of $\theta - \mathcal{L}$ - fuzzy metric space, we have $M(x_1, x_n, \theta(E_{\lambda,M}(x_1, x_2) + \dots + E_{\lambda,M}(x_{n-1}, x_n), n\delta)) \ge_L$

$$\mathcal{T}^{n-1}(M(x_1, x_2, \theta(E_{\lambda,M}(x_1, x_2), \delta),$$
$$M(x_2, x_3, \theta(E_{\lambda,M}(x_2, x_3), \delta),$$
$$\ldots M(x_{n-1}, x_n, \theta(E_{\lambda,M}(x_{n-1}, x_n), \delta))$$
$$\geq_L \mathcal{T}^{n-1}((N(\lambda), \ldots, N(\lambda)) \geq_L N(\mu)$$
$$E_{\mu,M}(x_1, x_n) \leq_L \theta(E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \ldots + E_{\lambda,M}(x_{n-1}, x_n), n\delta).$$

for eveiy $\delta > 0,$ which implies that

Since $\delta > 0$ is arbitrary, we have

J '

$$E_{\mu,M}(x_1, x_n) \leq_L \theta(E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \ldots + E_{\lambda,M}(x_{n-1}, x_n))$$
$$\leq_L E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \ldots + E_{\lambda,M}(x_{n-1}, x_n).$$

This proves (i). For (ii), note that since M is continuous in its third place, $E_{\lambda,M}(x, y)$ is not an element of the set $\{t > 0 : M(x, y, t) >_L N(\lambda)\}$ as soon as $x \neq y.$ Hence we have, $M(x_n, x, \eta) >_L N(\lambda) \Leftrightarrow E_{\lambda,M}(x_n, x) < \eta$ for eveiy $\eta > 0.$ This completes the proof.

Lemma 2.4. Let *A, B. S. T. I* and *J* be mappings from $\theta - \mathcal{L}$ - fuzzymetric space $(X, M, \mathcal{T})$ into itself satisfying

(i) $AI(X) \subset T(X), BJ(X) \subset S(X)$

(ii)
$$\int_0^{M(AIx, BJy, \varphi(t))} \phi(s)\,ds \geq_L r\left(\int_0^{M(x, y, t)} \phi(s)\,ds\right)$$

$$M(x, y, t) = \max\{M(Sx, Ty, t), M(AIx, Sx, t), M(BJy, Ty, t), [M(AIx, Ty, t) + M(BJy,$$

where $\phi(t)$ satisfies condition (ϕ) and $r : L \to L$ is a continuous function such that $r(t) >_L t$ for each $t = (t_1, t_2) \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and for all $x, y \in X.$ Then the sequence $\{y_n\}$ defined by $y_{2n} = Tx_{2n+1} = AIx_{2n}, y_{2n+1} = Sx_{2n+2} = BJx_{2n+1}, n = 0, 1, 2, \ldots$ is a Cauchy sequence in X Proof. For $t > 0,$ we have,

$$M(y_{2n}, y_{2n+1}, \varphi^n(t)) = M(AIx_{2n}, BJx_{2n+1}, \varphi^n(t)) \geq_L r\max\{M(Sx_{2n}, Tx_{2n+1}, \varphi^n(t)),$$
$$M(AIx_{2n}, Sx_{2n}, \varphi^n(t)), M(BJx_{2n+1}, Tx_{2n+1}, \varphi^n(t)),$$
$$[M(AIx_{2n}, Tx_{2n+1}, \varphi^n(t)) + M(BJx_{2n+1}, Sx_{2n}, \varphi^n(t))]/2\}$$
$$= r(M(y_{2n-1}, y_{2n}, \varphi^n(t))$$
$$\geq_L M(y_{2n-1}, y_{2n}, \varphi^n(t))$$
$$\geq_L M(y_{2n-2}, y_{2n-1}, \varphi^{n-1}(t)) \ldots \geq_L M(y_0, y_1, t)$$

for $n = 1, 2, \ldots$ which implies that, for eveiy $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\},$ we have

$$E_{\lambda,M}(y_{2n}, y_{2n+1}) = \inf\{\varphi^n(t) > 0 : M((y_{2n}, y_{2n+1}, \varphi^n(t)) >_L N(\lambda)\}$$
$$= \inf\{\varphi^n(t) > 0 : M((y_0, y_1, t)) >_L N(\lambda)\}$$
$$= \varphi^n(t)\{\inf\{t > 0 : M((y_0, y_1, t)) >_L N(\lambda)\}$$
$$= \varphi^n(t) E_{\lambda,M}(y_0, y_1)$$

From Lemma 2.3, for eveiy $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that

$$E_{\mu,M}(y_n, y_m) \leq_L E_{\lambda,M}(y_n, y_{n+1}) + E_{\lambda,M}(y_{n+1}, y_{n+2}) + \ldots + E_{\lambda,M}(y_{m-1}, y_m)$$
$$\leq \sum_{j=n}^{m-1} \varphi^j(E_{\lambda,M}(y_0, y_1)) \to 0 \text{ as } m, n \to \infty.$$

Thus $\{y_n\}$ is a Cauchy sequence in X. Now we prove the following common fixed point theorem.

Theorem 2.4. Let *A, B. S. T. I* and J be mappings from a complete $\theta - \mathcal{L}$-fuzzy metric space $(X, M, \mathcal{T})$ into itself satisfying (i), (ii) of Lemma 2.4. and property $(C).$ Suppose that one of *A. B. S, T. I* and J is complete and pairs (AI, S) and (BJ, T) are weakly compatible, then *A, B. S. T. I* and J have a unique common fixed point.

Proof. By Lemma 2.4. $\{y_n\}$ is a Cauchy sequence and since A'is complete, therefore, $\{y_n\}$ converges to some point $z \in X.$ Consequently, the subsequences $\{AIx_{2n}\}, \{Sx_{2n+2}\}, \{BJx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to $z.$

Assume that $S(X)$ is complete, so there exists a point u in *X* such that $Su = z.$ If $z \neq AIu,$ from (ii), we have

$$\int_0^{M(AIu, BJx_n, \varphi^n(t))} \phi(s)\,ds \geq_L r\left(\int_0^{M(u, x_n, t)} \phi(s)\,ds\right)$$

$$M(u, x_n, t) = \max\{M(Su, Tx_n, t), M(AIu, Su, t), M(BJx_n, Tx_n, t), [M(AIu, Tx_n, t) + M(BJx_n, Su)]/2\}$$

On the other hand, by Lemma 2.2, $M(AIu, BJx_n, \varphi^n(t)) \leq_L M(AIu, BJx_n, t).$

Taking the limit, $n \to \infty$ we get,

$$M(u, z, t) = \max\{M(z, z, t), M(AIu, z, t), M(z, z, t), [M(AIu, z, t) + M(z, z, t)]/2\}$$

$$\int_0^{M(AIu, z, t)} \phi(s)\,ds \geq_L r\left(\int_0^{M(AIu, z, t)} \phi(s)\,ds\right)$$

which is a contradiction. Thus we have. $AIu = Su = z.$

Since $AI(X) \subset T(X)$, there exists $v \in X$, such that $Tv = z$.

If $z \neq BJv$, we have

$$\int_0^{M(z,\,BJv,\,\varphi^n(t))} \phi(s)\,ds = \int_0^{M(AIu,\,BJv,\,\varphi^n(t))} \phi(s)\,ds \geq_L r\left(\int_0^{M(u,\,v,t)} \phi(s)\,ds\right)$$

where,

$$M(u,\,v,\,t) = \max\{M(Su,\,Tv,\,t),\,M(AIu,\,Su,\,t),\,M(BJv,\,Tv,\,t),\,[M(AIu,\,Tv,\,t) + M(1$$

But by Lemma 2.2, $M(AIu,\,BJv,\ \varphi^n(t)) \leq_L M(AIu,\,BJv,\ t)$. Thus we have,

$$\int_0^{M(z,\,BJv,\,t)} \phi(s)\,ds \geq_L r\left(\int_0^{M(z,\,BJv,t)} \phi(s)\,ds\right)$$

Hence. $Tv = BJv = AIu = Su = z$.

Since the pairs $(AI,\,S)$ and (BJ,T) are weakly compatible, we have $AISu = SAIu$, and $BJTv = TBJv$; i.e., $AIz = Sz$ and $BJz = Tz$.

If $AIz \neq z$, and

$$M(AIz,\,BJv,\,t) = \max\{M(Sz,\,Tv,\,t),\,M(AIz,\,Sz,\,t),\,M(BJv,\,Tv,\,t),\,[M(AIz,\,Tv,\,t) + M(BJv,\,Sz,\,t)]/2\}$$

or.

$$M(AIz,\,BJv,\,t) = \max\{M(AIz,\,z,\,t),\,M(AIz,\,AIz,\,t),\,M(z,\,z,\,t),\,[M(AIz,\,z,\,t) + M(z,\,AIz,\,t)]/2\}$$

We get.

$$\int_0^{M(AIz,\,z,\,\varphi^n(t))} \phi(s)\,ds = \int_0^{M(AIz,\,BJv,\,\varphi^n(t))} \phi(s)\,ds \geq_L r\left(\int_0^{M(AIz,\,z,\,t)} \phi(s)\,ds\right) >_L \int_0^{M(AIz,\,z,\,t)} \phi(s)\,ds$$

By Lemma 2.2, $M(AIz,\,z,\ \varphi^n(t)) \leq_L M(AIz,\,z,\ t)$, so we have

$$\int_0^{M(AIz,\,z,\,t)} \phi(s)\,ds \geq_L \int_0^{M(AIz,\,z,\,t)} \phi(s)\,ds,$$

which is a contradiction. Hence $AIz = z = Sz$. Similarly, we can prove that $BJz = z = Tz$. Therefore, z is a common fixed point of *A. B. S, T. I* and *J.*

For uniqueness, let if possible $z' \neq z$ be another common fixed point of *A. B.* .S', *T. I* and *J.*

Then there exists $t > 0$ such that $M(z,\ z',\ \varphi^n(t)) <_L 1$ and

$$\int_0^{M(z,\,z',\,\varphi^n(t))} \phi(s)\,ds = \int_0^{M(AIz,\,BJz',\,\varphi^n(t))} \phi(s)\,ds$$

But,

$$M(AIz,\,BJz',\,t) = \max\{M(Sz,\,Tz',t),\,M(AIz,\,Sz,\,t),\,M(BJz',\,Tz',\,t),\,[M(AIz,\,Tz',\,t) + M(BJz',\,Sz)]/2\}$$
$$= M(z,\,z',t)$$

Thus,

$$\int_0^{M(AIz,\,BJz',\,\varphi^n(t))} \phi(s)\,ds \geq_L \int_0^{M(z,\,z',\,\varphi^{n-1}(t))} \phi(s)\,ds \geq_L \ldots$$
$$\geq_L r\left(\int_0^{M(z,\,z',\,t)} \phi(s)\,ds\right) \geq_L \int_0^{M(z,\,z',\,t)} \phi(s)\,ds$$

By Lemma 2.2, $M(z,\ z',\ \varphi^n(t)) \leq_L M(z,\ z',\ t)$.

Hence, $M(z,\ z',\ t) = C$ for all $t > 0$. Since, M has the property (C), it follows that $C = 1_{\mathcal{L}}$, therefore $z = z'$, i.e., z is a unique common fixed point of *A, B. S. T. I* and *J.* This completes the proof

If we put $B = S = I = J = id$, i.e., the identity map and $\phi(t) = kt,\ 0 < k < 1,\ \ \varphi(t) = 1$ and $\theta(s,\,t) = s + t$, 111 Theorem 2.4. then the following result of Mamo et al (2012) is deduced.

Corollary 2.3.. Let *A* and *T* be mappings from a complete fuzzy metric space $(X,\,M,\,\mathcal{T})$ into itself satisfying $A(X) \subset T(X)$ and $M(Ax,\,Ay,\,kt) \geq M(Tx,\,Ty,\,t)$. Suppose that either $A(X)$ or $T(X)$ is complete and *A* and *T* are weakly compatible on *X.* Then *A* and *T* have a unique common fixed point.

In the following Theorem we relax the condition of completeness on the space.

Theorem 2.5. Let *A. B. S*, *T. I* and *J* be mappings from a $\theta - \mathcal{L}$- fuzzy metric space $(X,\,M,\,\mathcal{T})$ into itself satisfying (i), (ii) of Lemma 2.4 and property (C). Suppose that one of *A*, *B. S. T. I* and *J* is complete and pairs $(AI,\,S)$ and $(BJ,\,T)$ are weakly compatible, then *A. B. S. T. I* and *J* have a unique common fixed point.

Proof. From the proof of Theorem 2.4, we conclude that $\{y_n\}$ is a Cauchy sequence in A'.

Assume that $S(X)$ is complete subspace of *X.* Then the subsequence of $\{y_n\}$ must get a limit in $S(X)$. Let it be u and $Sv = u$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of subsequence of the convergent sequence. Now we have,

$$\int_0^{M(AIv,\,BJx_n,\,\varphi^n(t))}\phi(s)\,ds \geq_L r\left(\int_0^{M(v,\,x_n,\,t)}\phi(s)\,ds\right)$$

where,

$$M(v,x_n,t)=\max\{M(Sv,Tx_n,t),\,M(AIv,Sv,t),\,M(BJx_n,Tx_n,t),\,[M(AIv,Tx_n,t)+M(BJx_n,Sv)]/2\}$$

Now by Lemma 2.2, $M(AIv,\,BJx_n,\,\varphi^n(t)) \leq_L M(AIv,\,BJx_n,\,t).$

Taking the limit $n\to\infty$, we get,

$$M(v,u,t)=\max\{M(u,u,t),\,M(AIv,Sv,t),\,M(u,u,t),\,[M(AIv,u,t)+M(u,u)]/2\}$$

$$\int_0^{M(AIv,\,u,\,t)}\phi(s)\,ds \geq_L r\left(\int_0^{M(AIv,\,u,\,t)}\phi(s)\,ds\right),$$

which is a contradiction. Thus we have, $AIv=Sv=u,$ which shows that pair $(AI,\,S)$ has a point of coincidence.

Since $AI(X)\subset T(X),$ there exists $p\in X,$ such that $Tp=u.$

If $u\neq BJp,$ we have

$$\int_0^{M(u,\,BJp,\,\varphi^n(t))}\phi(s)\,ds = \int_0^{M(AIv,\,BJp,\,\varphi^n(t))}\phi(s)\,ds$$

We have,

$$\begin{aligned}M(AIv,BJp,t) &= \max\{M(Sv,Tp,t),\,M(AIv,Sv,t),\,M(BJp,Tp,t),\,[M(AIv,Tp,t)+M(BJp,Sv,t)]/2\}\\ &=\max\{M(u,u,t),\,M(u,u,t),\,M(BJp,u,t),\,[M(u,u,t)+M(BJp,u,t)]/2\}\end{aligned}$$

By Lemma 2.2, $M(AIv,\,BJp,\,\varphi^n(t)) \leq_L M(AIv,\,BJp,\,t).$
Thus we have,

$$\int_0^{M(u,\,BJp,\,t)}\phi(s)\,ds \geq_L r\left(\int_0^{M(u,\,BJp,\,t)}\phi(s)\,ds\right)$$

a contradiction. Hence, $Tp=BJp=AIv=Sv=u.$

Since the pairs $(AI,\,S)$ and (BJ,T) are weakly compatible, we have $AISv=SAIv,$ and $BJTp=TBJp,$ i.e., $AIu=Su$ and $BJu=Tu.$

If $AIu\neq u,$ and

$$\begin{aligned}M(AIu,BJp,t) &= \max\{M(Su,Tp,t),\,M(AIu,Su,t),\,M(BJp,Tp,t),\,[M(AIu,Tp,t)+M(BJp,Su,t)]/2\}\\ &=\max\{M(AIu,u,t),\,M(AIu,AIu,t),\,M(Tp,Tp,t),\,[M(AIu,u,t)+M(u,AIu,t)]/2\}\end{aligned}$$

We set

$$\int_0^{M(AIu,\,u,\,\varphi^n(t))}\phi(s)\,ds = \int_0^{M(AIu,\,BJp,\,\varphi^n(t))}\phi(s)\,ds \geq_L r\int_0^{M(AIu,\,BJp,\,\varphi^n(t))}\phi(s)\,ds >_L \int_0^{M(AIu,\,u,\,t)}\phi(s)\,ds$$

By Lemma 2.2, $M(AIu,\,u,\,\varphi^n(t))\leq_L M(AIu,\,u,\,t),$ so we have

$$\int_0^{M(AIu,\,u,\,t)}\phi(s)\,ds \geq_L \int_0^{M(AIu,\,u,\,t)}\phi(s)\,ds,$$

which is a contradiction. Hence $AIu=u=Su.$ Similarly, we can prove that $BJu=u=Tu.$ Therefore *it* is a common fixed point of *A*, *B.* S, *T*, *I* and *J.*

For uniqueness, let if possible $u'\neq u$ be another common fixed point of *A*, *B. S. T*, *I* and *J.*

Then there exists $t>0$ such that $M(u,u',\varphi^n(t))\leq_L$ 1 and

$$\int_0^{M(u,\,u',\,\varphi^n(t))}\phi(s)\,ds \geq_L \int_0^{M(AIu,\,BJu',\,\varphi^n(t))}\phi(s)\,ds.$$

But,

$$\begin{aligned}M(AIu,\,BJu',\,t)=\max\{&M(Su,\,Tu',\,t),\,M(AIu,\,Su,\,t),\,M(BJu',\,Tu',\,t),\\ &[M(AIu,\,Tu',\,t)+M(BJu',\,Su,\,t)]/2\}=M(u,\,u',\,t)\end{aligned}$$

Thus,

$$\int_0^{M(AIu,BJu',\,\varphi^n(t))}\phi(s)\,ds \geq_L \int_0^{M(u,u',\,\varphi^{n-1}(t))}\phi(s)\,ds \geq_L \cdots$$
$$\geq_L r\left(\int_0^{M(u,u',t)}\phi(s)\,ds \geq_L\right) \geq_L \int_0^{M(u,u',t)}\phi(s)\,ds.$$

By Lemma 2.2, $M(u,\,u',\,\varphi^n(t))\leq_L M(u,\,u',\,t).$

Hence, $M(u,\,u',\,t)=C$ for all $t>0.$ Since. *M* has the property $(C),$ it follows that $C=1_{\mathcal{L}},$ therefore $u'=u,$ i.e., u is a unique common fixed point of *A. B.* S, *T*, *I* and *J.* This completes the proof.

If we put $B=S=I=J=$ identity map and $\phi(t)=kt,\ 0<k<1,\ \varphi(t)=1$ and $\theta(s,t)=s+t,$ in Theorem 2.5, then the following result of Manro et al is obtained.

Corollary 2.3. Let *A* and *T* be mappings from a fuzzy metric space $(X,M,\mathcal{T})$ into itself satisfying $A(X)\subset T(X)$ and $M(Ax,\,Ay,\,kt)\geq M(Tx,\,Ty,\,t).$ Suppose that either $A(X)$ or $T(X)$ is complete and *A* and *T* are wejakly compatible on *X.* Then *A* and *T* have a unique common fixed point.

Theorem 2.6. All the statements of the Theorem 2.4 remains true if a weakly compatible property is replaced by any one of the following:

(i) 5-weakly commuting property,

(ii) 5-weakly commuting property of type (A_f),

(iii) 5-weakly coimnuting property of type (A_g),

(iv) weakly coimnuting property.

Proof, (i) Since all the conditions of Theorem 2.4 are satisfied, the existence of coincidence points for both the pairs (AI, S) and (BJ,T) are ensured. Let x be an arbitrary point of coincidence for the pair (AI, S) and (BJ,T). Then using 5-weakly commutativity, one gets

$$M(AISx,\ SAIx,\ \varphi^n(t)) \geq_L M(AIx,\ Sx,\ t/R) = M(Sx,\ Sx,\ t/R) = 1_L$$

and

$$M(BJTx,\ TBJx,\ \varphi^n(t)) \geq_L M(BJx,\ Tx,\ t/R) = M(Tx,\ Tx,\ t/R) = 1_L$$

From Lemma 2.2,

$$M(AISx,\ SAIx,\ \varphi^n(t)) \leq_L M(AISx,\ SAIx,\ t)$$

and

$$M(BJTx,\ TBJx,\ \varphi^n(t)) \leq_L M(BJTx,\ TBJx,\ t)$$

So we have $AISx = SAIx$ and $BJTx = TBJx.$ Thus the pairs (AI, S) and (BJ,T) are weakly compatible. Now applying Theorem 2.4, we conclude that *A, B, S. T. I* and *J* have a unique conunon fixed point.

(ii) In case pair (AI, S) satisfying property 5-weakly commutativity of type (A_f), we have $M(SSx,\ AISx,\ \varphi^n(t)) \geq_L M(Sx,\ AIx,\ t/R) = M(AIx,\ AIx,\ t/R) = 1_L$ and we have $AISx = SSx.$

From Lemma 2.2, $M(AISx,\ SAIx,\ \varphi^n(t)) \leq_L M(AISx,\ SAIx,\ t).$ So,

$$M(AISx,\ SAIx,\ t) \geq_L \mathcal{T}(M(AISx,\ SSx,\ t/2),\ M(SSx,\ SAIx,\ t/2))$$
$$= \mathcal{T}(M(AISx,\ AISx,\ t/2),\ M(x,\ x,\ t/2)) \geq_L \mathcal{T}(1_L, 1_L) = 1_L.$$

This implies that $AISx = SAIx.$

Similarly, if pair (BJ,T) satisfies property 5-weakly commutativity of type (A_f), we get $BJTx = TBJx.$

(iii) In case pair (AI, S) satisfying property 5-weakly commutativity of type (A_g), we have $M(AIAIx,\ SAIx,\ \varphi^n(t)) \geq_L M(AIx,\ Sx,\ t/R) = M(AIx,\ AIx,\ t/R) = 1_L$ and we have $AIAIx = SAIx.$

From Lemma 2.2, $M(AISx,\ SAIx,\ \varphi^n(t)) \leq_L M(AISx,\ SAIx,\ t).$

$$M(AISx,\ SAIx,\ t) \geq_L \mathcal{T}(M(AISx,\ AIAIx,\ t/2),\ M(AIAIx,\ SAIx,\ t/2))$$

So, $$= \mathcal{T}(M(x,\ x,\ t/2),\ M(AIAIx,\ AIAIx,\ t/2)) \geq_L \mathcal{T}(1_L, 1_L) = 1_L.$$

This implies that $AISx = SAIx.$

Similarly, if pair (BJ,T) satisfies property 5-weak commutativity of type (A_f), we get $BJTx = TBJx.$

Similarly, we can show that, if pairs (AI, S) and (BJ,T) are weakly commuting then (AI, S) and (BJ,T) also commutes at their point of coincidence. Now, in view of Theorem 2.4. in all five cases, *A.* B, *S. T. I* and *J* have a unique common fixed point. This completes the proof.

CONCLUSION

In this examination, we have displayed the most critical fixed point theorems in the expository investigation of problems in the connected science: Banach's and Schauder's fixed point theorem. In our unique situation, Brouwer's fixed point might be considered as a result which empowers us to proof the Schauder fixed point theorem. The fixed point theorem of Darbo is the focal result of a bound together approach for Banach's and Schauder's theorem.

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Corresponding Author

Seema Devi*

Research Scholar, OPJS University, Churu, Rajasthan

E-Mail – arora.kips@gmail.com