Journal of Advances and Scholarly Researches in Allied Education
Vol. 13, Issue No. 1, April-2017, ISSN 2230-7540

A Research on Some Decompositions Strategies of Function Algebras and Function Spaces: A Review

Kusum Devi[1]* Dr. Sudesh Kumar[2]

[1] Research Scholar of OPJS University, Churu, Rajasthan

[2] Professor, OPJS University, Churu, Rajasthan

Abstract – "The only way to learn mathematics is to do mathematics." Halmos is certainly not alone in this belief. The current set of notes is an activity-oriented companion to the study of linear functional analysis and operator algebras. It is intended as a pedagogical companion for the beginner, an introduction to some of the main ideas in this area of analysis, a compendium of problems I think are useful in learning the subject, and an annotated reading/reference list. The great majority of the results in beginning functional analysis are straightforward and can be verified by the thoughtful student.

---------------------------------X---------------------------------

INTRODUCTION

Different determinations of "variable based math" have been given by various journalists. The main notice of the word is to be found in the title of a book "Hidab al-jabrwal-muqubala" written in Baghdad around 825 A.D. by the Arab mathematician Mohammed ibn-Musa al-Khowarizmi. The words jabr (JAH-ber) and muqubalah (moo-KAH-ba-lah) were utilized by al-Khowarizmi to assign two essential tasks in fathoming conditions. Jabr was to transpose subtracted terms to the opposite side of the condition. Muqubalah was to drop like terms on inverse sides of the condition. Truth be told, the title has been meant signify "study of reclamation and resistance" or "study of transposition and scratch-off" and "The Book of Completion and Cancellation" or "The Book of Restoration and Balancing". Jabr is utilized in the progression where x - 3 = 10 moves toward becoming x = 13. The left-half of the principal condition, where x is decreased by 3, is "reestablished" or "finished" back to x in the subsequent condition. Muqabalah takes us from x + y = y + 4 to x = 4 by "dropping" or "adjusting" the different sides of the condition. In the long run the muqabalah was abandoned, and this sort of math wound up known as polynomial math in numerous dialects.

Different essayists have gotten the word from the Arabic molecule al (the distinct article), and gerber, signifying "man." Since, notwithstanding, Geber happened to be the name of an observed Moorish rationalist who prospered in about the eleventh or twelfth century, it has been assumed that he was the organizer of polynomial math, which has since propagated his name. The proof of Peter Ramus on this point is intriguing, yet he gives no expert for his particular articulations. In the prelude to his Arithmeticaelibri pair ettotidem Algebrae (1560), he says: "The name Algebra is Syriac, implying the workmanship or precept of a magnificent man. For Geber, in Syriac, is a name connected to men, and is in some cases a term of respect, as ace or specialist among us. The expression "variable based math" is presently in all inclusive use.

Early Indian and Chinese geometrical issues included mathematical conditions and their answers like those of the Greeks who tackled numerous nearly troublesome logarithmic issues in a simply geometrical manner. While the Greek variable based math was created by Diophantus in his Arithmatica, in the third century A. D., the variable based math in Babylon was grown a lot before in a further developed structure including issues on cubic and biquadratic conditions as appeared by Neugebauer and others. Once can't resist pondering whether this Babylonian polynomial math could have been transmitted in original structures to establish the framework of Indian and Chinese variable based math from one viewpoint and for the Hellenistic improvement on the other. During the rot of Western Science in the early Middle Age, the polynomial math of the Diophantine time frame was overlooked and when the incomparable Arab Scientific Movement occurred, Arabic variable based math in all respects presumably got its motivation from India as opposed to from Greece.

In India, the geometrical techniques for tackling mathematical issues have been followed to the different Sulba Sutras. The Shulba Sutras are a piece of the bigger corpus of writings called the Shrauta Sutras viewed as reference sections to the Vedas. They are the main wellsprings of information of Indian science from the Vedic time frame. The four noteworthy Shulba Sutras, which are numerically the most critical, are those formed by Baudhayana, Manava, Apastamba and Katyayana, about whom next to no is known. Pythagoras hypothesis and Pythagorean triples, as found in the Sulba Sutras. The rope extended along the length of the slanting of a square shape makes a territory which the, vertical and level sides make together, as such: a2 = b2 + c2. Instances of Pythagorean triples given as the sides of right angled triangles:

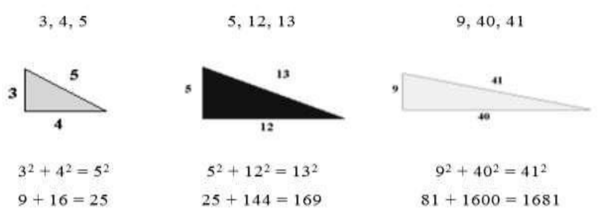

Figure 1.1 Right angled triangles

These incorporate arrangements of direct, synchronous and even vague conditions, emerged regarding the development of various kinds of conciliatory special raised areas and game plans for laying blocks into them. In the advancement of early science, when the images for activity started to be utilized in the calculations another branch developed being isolated from math and geometry which is known as variable based math. The separation of variable based math as an unmistakable branch from arithmetic by and large took Introducation Algebraic comprehending capacity had been examined by numerous instructors and analyst. A large portion of their perspectives concentrated on recognizing the sorts of arithmetical procedures which are required in illuminating logarithmic undertakings or procedure based.

There exists no unequivocal definition for logarithmic settling capacity as it very well may be seen from alternate points of view. A mathematician's perspective on logarithmic explaining capacity isn't generally equivalent to the perspective on an analyst, a primary teacher or a specialist on arithmetical illuminating capacity. In this way, the topic of 'what is mathematical settling capacity?' isn't the sort of inquiry that is promptly replied by exact research. In any case, we give a few points of view of the idea of logarithmic settling capacity if the inquiry is reworded: What sorts of mathematical procedures that show the capacity in taking care of arithmetical issue? Kept up that the capacity of utilizing mathematical condition to take care of and speak to the issue circumstance includes various arithmetical procedures which comprise of three stages, specifically: I) examining the example by gathering the numerical information; ii) speaking to and summing up the example into a table and a condition; and iii) deciphering and applying the condition to the related or new circumstance. These three periods of mathematical procedure are depicted in some detail underneath. Three periods of mathematical procedures In the principal stage, understudies will be given a progression of assignments including explicit cases.

The understudies are relied upon to have the option to see and perceive the theme while working with the numerical models. The reactions of understudies are assortment while taking care of this kind of issue because of the various degrees of logarithmic illuminating capacity and points of view. In the subsequent stage, understudies at that point might decide whether they would speak to their numerical information into a table which is an ordinarily utilized type of mathematical portrayal. The portrayal gives a perception of two related amounts (autonomous variable and ward variable) and it can assist them with finding the example. tables are efficient portrayals for a progression of explicit cases. It can give understudies a feeling of the dynamic connection between the factors while they speak to information in table. Henceforth, regarding portrayal, the degrees of comprehension of example among the understudies can be controlled by educator. Next, understudies are required to sum up the relationship in the issue circumstance emblematically utilizing logarithmic condition. Making speculation through some particular cases is one of the significant discernments to express all inclusive statement in an issue circumstance.

Understudies are most likely observing the example through the specific number, the specific calculation and mindful of sweeping statement. Noticed that when understudies are stood up to with 'disagreeable' or enormous number of explicit model, it will push them to make a speculation for the example and they brief to give the reactions without seeing or draw them all. In the last stage, understudies could decide whether and how they test the guess by applying the standard into the comparative or new circumstance. In this procedure, understudies are required to translate and apply the condition to tackle the related or new issue circumstance so as to legitimize their decisions. As indicated by procedure of testing guess serves to create deductive thinking process. It decides the legitimate outcome of the presumption or guess that the understudies made. Along these lines, testing guess takes into account significant utilization of logarithmic control as a major aspect of higher mathematical unraveling capacity.

Place, from about the hour of the Brahmagupta (598 A.D.), following the method of uncertain examination. Truth be told, Brahmagupta utilized the term kuttaka–ganita or basically kuttaka for variable based math. The term kuttaka signifying "pummel alludes to a part

www.ignited.in

of the study of variable based math managing the subject of straight condition, quadratic condition and vague conditions of first, second and higher degree. It is intriguing to find that this subject was considered so significant by the Hindus that the entire study of variable based math was named after it in the start of the seventh century." Algebra is likewise called avyakta - ganita or "the study of estimation with questions" (avyakta = obscure) in contradistinction to the study of computation with known (vyakta = known) for number juggling including geometry and mensuration. The term bijaganita meaning study of figuring with components or obscure amounts (bija) was implied by Prthudakasvami (850 A. D.) and utilized with definition by Bhaskara - II (1150 A.D.).

It is generally recognized that variable based math is a fundamental piece of undergrad numerical learning but it is likewise known for its abnormal state of trouble at the university level. Numerous undergrad and graduate understudies, including forthcoming instructors, battle to handle even the most central ideas of variable based math. For the majority of the understudies experience scientific reflection and formal verification. Presently, it is frequently the first run through in which instructors anticipate that understudies should "go past learning 'imitative personal conduct standards' for copying the arrangement of an enormous number of minor departure from few issues 1by expecting confirmations to clarify conceptual hypotheses and thoughts. Specifically, understudies are relied upon to rationally build new items dependent on a rundown of properties and afterward work on these articles. In any case, essentially being presented to these unique ideas does not infer the advancement of scientific significance. Understudies must play a functioning job in the learning procedure by structure on their past numerical information to comprehend conceptual ideas.

Cook (2012) affirmed in his exposition that the trouble understudies involvement in dynamic variable based math is because of the absence of built up associations between undergrad science and school arithmetic. He avowed that forthcoming instructors "don't expand upon their rudimentary understandings of variable based math, leaving them incapable to convey hints of any profound and binding together thoughts that administer the subject". These guesses suggest that undergrad educators must probably pass on a unique plan to understudies as well as give understudies the chance to fabricate numerical significance upon these deliberations. On the off chance that instructors don't have the foggiest idea how to make an interpretation of those reflections into a structure that empowers students to relate the arithmetic to what they definitely know, they won't learn with comprehension. Consequently, we can just anticipate that college understudies should truly get to the advantages of this investigation through complete appreciation by interfacing theoretical hypothesis to past learning and thoughts to help in the development of numerical importance.

THE EDUCATIONAL LEVEL OF THE LEARNER BEAR UPON THE ROLE OF HISTORY OF MATHEMATICS OR ALGEBRA

The way history of mathematics can be used, and the rationale for its use, may vary according to the educational level of the class: children at elementary school and students at university do have different needs and possibilities. Questions arise about the ways in which history can address these differences. This may, again, be reflected in different training needs for teachers at these levels. To speak about the ``use'' of the history of mathematics stand out that history of mathematics is something external to mathematics. This assumption would not be universally agreed, however.

HISTORY OF MATHEMATICS/ALGEBRA AS A TAUGHT SUBJECT BECOME RELEVANT

In dissecting the job of history of science, it is imperative to recognize issues around utilizing history of arithmetic in a circumstance whose prompt reason for existing is the instructing of math, and showing the historical backdrop of arithmetic all things considered, in a course or a shorter session. It may be the case that courses throughout the entire existence of science, and its study hall use, ought to be incorporated into an educator preparing educational program. There is likewise a third zone, related yet discrete, to be specific the historical backdrop of arithmetic training, which is a fairly unique sort of history.

THE PARTICULAR FUNCTIONS OF A HISTORY OF MATHEMATICS COURSE OR COMPONENT FOR TEACHERS

History of arithmetic may assume a particularly significant job in the preparation of future instructors, and furthermore educators experiencing in-administration preparing. There are various purposes behind incorporating a recorded segment in such preparing, including the advancement of energy for science, empowering students to see understudies in an unexpected way, to see arithmetic in an unexpected way, and to create aptitudes of perusing, library use and explanatory composition which can be disregarded in science courses. It might be valuable here to separate the preparation requirements for essential, optional and higher levels.A related issue is the thing that sorts of history of arithmetic is suitable in educator preparing and why: for instance, it may be the case that the historical backdrop of the establishments of science and thoughts of thoroughness and evidence are particularly significant for future auxiliary and tertiary instructors. This issue is additionally important for different classes than future educators, and is gotten again being referred to 5.

DIFFERENT PARTS OF THE CURRICULUM INVOLVE HISTORY OF MATHEMATICS IN A DIFFERENT WAY

Already research is taking place to investigate the particularities of the role of history in the teaching of algebra, compared with the role of history in the teaching of geometry. Different parts of the syllabus make reference, of course, to different aspects of the history of mathematics, and it may be that different modes of use are relevant. Looking at the curriculum in a broad way, we may note that the histories of computing, of statistics, of core "pure" mathematics and of the interactions between mathematics and the world are all rather different pursuits. Even for the design of the curriculum historical knowledge may be valuable. A survey of recent trends in research could lead to suggestions for new topics to be taught.

HISTORY OF MATHEMATICS PLAY IN SUPPORTING SPECIAL EDUCATIONAL NEEDS

The experience of teachers with responsibility for a wide variety of special educational needs is that history of mathematics can empower the students and valuably support the learning process. Among such areas are experiences with mature students, with students attending numeracy classes, with students in particular apprenticeship situations, with hitherto low-attaining students, with gifted students, and with students whose special needs arise from handicaps. Here the many different experiences need to be researched, their particular features drawn out, and an account provided in an overall framework of analysis and understanding.

ETYMOLOGY

The word "algebra" is derived from the Arabic word الجبر al-jabr, and this comes from the treatise written in the year 830 by the medieval Persian mathematician, Muhammad ibn Mūsā al-Khwārizmī, whose Arabic title, Kitāb al-muḫtaṣar fī ḥisāb al-ğabr wa-l-muqābala, can be translated as The Compendious Book on Calculation by Completion and Balancing. The treatise provided for the systematic solution of linear and quadratic equations. According to one history, "[i]t is not certain just what the terms al-jabr and muqabalah mean, but the usual interpretation is similar to that implied in the previous translation. The word 'al-jabr' presumably meant something like 'restoration' or 'completion' and seems to refer to the transposition of subtracted terms to the other side of an equation; the word 'muqabalah' is said to refer to 'reduction' or 'balancing'—that is, the cancellation of like terms on opposite sides of the equation. Arabic influence in Spain long after the time of al-Khwarizmi is found in Don Quixote, where the word 'algebrista' is used for a bone-setter, that is, a 'restorer'."[1]The term is used by al-Khwarizmi to describe the operations that he introduced, "reduction" and "balancing", referring to the transposition of subtracted terms to the other side of an equation, that is, the cancellation of like terms on opposite sides of the equation.

ALGEBRAIC EXPRESSION

Variable based math did not generally utilize the imagery that is presently universal in science; rather, it experienced three unmistakable stages. The phases in the advancement of representative variable based math are around as follows: Logical variable based math, in which conditions are written in full sentences. For instance, the logical type of x + 1 = 2 is "The thing in addition to one equivalents two" or perhaps "The thing in addition to 1 equivalents 2". Expository variable based math was first created by the old Babylonians and stayed overwhelming up to the sixteenth century.

Syncopated variable based math, in which some imagery is utilized, however which does not contain the majority of the attributes of emblematic polynomial math. For example, there might be a limitation that subtraction might be utilized just once inside one side of a condition, which isn't the situation with representative variable based math. Syncopated mathematical articulation originally showed up in Diophantus' Arithmetica (third century AD), trailed by Brahmagupta's Brahma Sphuta Siddhanta (seventh century).

Emblematic polynomial math, in which full imagery is utilized. Early strides toward this can be found in crafted by a few Islamic mathematicians, for example, Ibn al-Banna (thirteenth fourteenth hundreds of years) and al-Qalasadi (fifteenth century), albeit completely representative polynomial math was created by François Viète (sixteenth century). Afterward, René Descartes (seventeenth century) presented the cutting edge documentation (for instance, the utilization of x—see underneath) and demonstrated that the issues happening in geometry can be communicated and unraveled as far as polynomial math .

Similarly significant as the utilization or absence of imagery in variable based math was the level of the conditions that were tended to. Quadratic conditions assumed a significant job in early variable based math; and all through the majority of history, until the early current time frame, every quadratic condition were delegated having a place with one of three classes.

- $x^2 + px = q$
- $x^2 = px + q$
- $x^2 + q = px$

where p and q are positive. This trichotomy comes about because quadratic equations of the form , with p and q positive, have no positive roots. In between the rhetorical and syncopated stages of symbolic algebra, a geometric constructive algebra was developed by classical Greek and Vedic Indian mathematicians in which algebraic equations were solved through geometry. For instance, an equation of the form was solved by finding the side of a square of area A.

DECOMPOSITIONS FOR SPECIAL VECTOR FUNCTION SPACES

Let A be a complex function space on X and B be a semisimple Banach algebra. It is natural to expect that the decompositions of A ~B and A are related. We show that they are, in fact, equal.

PROPOSITION

$$\mathscr{K}(A\hat{\otimes}B)=\mathscr{K}_E(A\hat{\otimes}B)=\mathscr{K}_E(A) \text{ and } \mathscr{F}(A\hat{\otimes}B)=\mathscr{F}_E(A\hat{\otimes}B)=\mathscr{F}_E(A).$$

Proof. The first part of both the equalities follows from Corollary 2.7. Also N(A~B)=N(A) as we have noted earlier. Therefore, oWE(AI~B)=X'E(A) and ~(A ~ B) = ~dA).

PROPOSITION

For a closed subset K of X, N (A ~ Bit) c N (A It). Consequently ~Fp(A) < ~f ~(A ~ B).

The decomposition theory. Let A be a C*-algebra with unit 1. In this section we shall always assume that A is uniformly separable. Let A* be the dual Banach space of A and S the set of all states on A; then S is oL4*,.4)-compact; let {a„} be a sequence of nonzero elements which is uniformly dense in the selfadjoint portion As of A. For cb,\l/eQ, define

$$d(\phi,\psi) = \sum_{i=1}^{\infty} \frac{|(\phi-\psi)(a_n)|}{2^n \| a_n \|};$$

The motivation behind this paper is to introduce some essential improvements associated with properties of capacity spaces characterized on limit spaces, rather than measure spaces. It is our inclination that these improvements, on account of their relations with significant parts of scientific investigation on one hand and their straightforward and fundamental character on the other, have the right to be generally known. The accentuation of our piece is set upon the investigation of the basic utilitarian logical components with the end goal that an attractive hypothesis can be created with regards to semi Banach spaces. One of the fundamental issues is that we are compelled to work with a non-added substance basic, the Choquet basic, so the double spaces are not effectively recognizable and some essential properties, for example, the overwhelmed intermingling hypothesis, are not longer accessible. In the writing, a limit on a space Ω is generally expected to be an expanding set capacity C : Σ → [0,∞], with Σ a group of subsets in Ω, with various properties relying upon the specific circumstance, and the Choquet necessary is characterized as

$$\int f\, dC := \int_0^{\infty} C\{f > t\}\, dt$$

on the off chance that f ≥ 0 is a quantifiable capacity as in {f > t} ∈ Σ for each t > 0. In numerous significant instances of limits the area Σ of C is a σalgebra. This is the situation of the variational limits, and of the Fuglede [18] and Meyers [21] limits of nonlinear potential hypothesis. They are countably subadditive set capacities on all subsets of Rn which incorporate the Riesz and the Bessel limits. In spite of the fact that they are not Caratheodory metric external measures, they fulfill a Fatou type condition and, by a general hypothesis because of G. Choquet (cf. [16, Chapter VI]), each Borel set B ⊂ Rn is capacitable, this implying

$$\sup\{C(K);\, K \subset B,\, K \text{ compact}\} = C(B) = \inf\{C(G);\, G \supset B,\, G \text{ open}\}.$$

Then the class of all Borel sets turns out to be a convenient domain for all of them. We refer to and for an extended overview of these capacities. Another well known class of capacities are the Hausdorff contents. If h is a continuous increasing function on [0,∞) vanishing only at 0, which is called a measure function. denote μh the corresponding Hausdorff measure on Rn , and let I or Ik represent a general cube in Rn with its sides parallel to the axes. The use of the corresponding Hausdorff capacity or Hausdorff content,

$$E_h(A) := \inf_{A \subset \bigcup_{k=1}^{\infty} I_k} \left\{ \sum_{k=1}^{\infty} h(|I_k|) \right\},$$

is often more convenient than μh, and Eh(A) = 0 if and only if μh(A) = 0. If h(t) = t α (α > 0), it is customary to write H∞ α instead of Eh, and this capacity is called the α-dimensional Hausdorff content. The case h(x) := x log(1/x) on [0, 1/e] corresponds to the Shannon entropy considered in [17]. New examples appear when studying interpolation properties of function spaces as in [15]. If E is a quasi-Banach function space on the measure space (Ω, Σ, μ), then

$$C_E(A) := \|\chi_A\|_E \qquad (A \in \Sigma)$$

defines a capacity and, as in the case of Hausdorff capacities, there is a measure μ such that $C_E(A) = 0$ 0 if and only if μ(A) = 0.

The goal of these notes is to clearly set the basic properties of the capacity spaces (Ω, Σ, C) and their associated Lebesgue spaces L p (C) and L p,q(C), to show how the general theory can be applied to function spaces such as classical Lorentz spaces, and to complete the real interpolation theory for these spaces started in [15] and [14].

Further applications of the use of these capacities will appear in forthcoming work. In [3] it will be shown how they are a useful tool to extend the Riesz-Herz estimates concerning the Hardy-Littlewood operator.

The notation $A \lesssim B$ means that $A \leq \gamma B$ for some absolute constant $\gamma \geq 1,$ and A $\simeq B$ if $A \lesssim B \lesssim A.$ A. We refer to [7] for general facts concerning function spaces.

Let (Ω, Σ) be a measurable space. Sets will always be assumed to be in the σ-algebra Σ and functions will be real mesurable functions on (Ω, Σ). From now on, by a capacity C we mean a set function defined on Σ satisfying at least the following properties:

(a) $C(\emptyset) = 0,$
(b) $0 \leq C(A) \leq \infty,$
(c) $C(A) \leq C(B)$ if $A \subset B$, and

(d) Quasi-subadditivity: $C(A \cup B) \leq c(C(A) + C(B)),$ where c ≥ 1 is a constant.

On the off chance that c = 1, we state that the limit is subadditive. In the event that C is a limit on Σ, we will say that (Ω, Σ, C) is a limit space. It will assume the job of a measure space (Ω, Σ, μ) in the hypothesis of Banach capacity spaces. We are going to check which of the properties for measure spaces are as yet fulfilled by limit spaces. The circulation work Cf and the nonincreasing adjustment f ? C are characterized as on account of measures by

$$C_f(t) := C\{|f| > t\},$$

And

$$f_C^\star(x) := \inf\{t;\, C\{|f| > t\} \leq x\} = \int_0^\infty \chi_{[0, C\{|f|>t\})}(x)\, dt,$$

Since $\{t;\, C\{|f| > t\} \leq x\}$ is the interval $[f_C^\star(x), \infty]$.

Many of the basic properties remain true in this capacitary setting. The following ones are easily proved:

(a) $(\chi_A)_C^\star = \chi_{[0,C(A))}.$
(b) If $s = \sum_{k=1}^N a_k \chi_{A_k}$, $A_k \cap A_j = \emptyset$ if $k \neq j$ and $a_1 > a_2 > \cdots > a_N > 0 = a_{N+1}$, then $s_C^\star = \sum_{k=1}^N (a_k - a_{k-1}) \chi_{[0, C(A_1 \cup \cdots \cup A_k))}.$
(c) If $\psi : [0, \infty) \to [0, \infty)$ is nondecreasing and right-continuous, then $\psi(|f|)_C^\star = \psi(f_C^\star)$. For instance, $(|f|^p)_C^\star = (f_C^\star)^p$ $(p > 0)$.

Note that

$$(f+g)_C^\star(x) \leq f_C^\star\Big(\frac{x}{2c}\Big) + g_C^\star\Big(\frac{x}{2c}\Big).$$

Indeed,let

$\lambda := f_C^\star(x_1) + g_C^\star(x_2) < \infty$ and $x_1, x_2 \geq 0$. Then

$$C\{|f+g| > f_C^\star(x_1) + g_C^\star(x_2)\} \leq cC_f(f_C^\star(x_1)) + cC_g(g_C^\star(x_2)) \leq cx_1 + cx_2,$$

In particular, $(f+g)_C^\star(x) \leq f_C^\star(x/2c) + g_C^\star(x/2c)$ as announced. A property is said to hold quasi-everywhere (C-q.e. for short) if the exceptional set has zero capacity.

Point wise convergence $f_n \to f$ will mean $C(\{f_n \not\to f\}) = 0.$

Similarly, fn ↑ f that fn → f and C({fn > fn+1}) = 0. Also, we write An ↑ A or An ↓ A when χAn ↑ χA or χAn ↓ χA in the above sense, respectively. If f ≥ 0, the Choquet integral

$$\int f\, dC := \int_0^\infty C\{f > t\}\, dt \in [0, \infty]$$

satisfies R f dC = 0 if and only if f = 0 C-q.e. and it is positive-homogeneous,

$$\int \alpha f\, dC = \alpha \int f\, dC \qquad (\alpha > 0).$$

Moreover, by Fubini's theorem

$$\int_0^\infty f_C^\star(x)\, dx = \int f\, dC.$$

The relation {f +g > t} ⊂ {f > t/2}∪ {g > t/2} shows that this integral, defined on nonnegative functions, is quasi-sub additive with constant 2c,

$$\int (f+g)\, dC \leq 2c\Big(\int f\, dC + \int g\, dC\Big).$$

Observe that, if f = g C-q.e. and C is subadditive, then R f dC = R g dC since, if A = {f 6= g}, then

$$C\{f > t\} \leq C\Big((\{f > t\} \cap A^c) \cup (\{f > t\} \cap A)\Big) \leq C\{g > t\}.$$

This will be also true if C(An) → C(A) whenever An ↑ A. In this case we say that C has the Fatou property (or that it is a Fatou capacity). If C is a Fatou

www.ignited.in

capacity, the countable unions of C−null sets are also

C−null. Indeed, C(A1∪· · ·∪An) ≤ c n (C(A1)+· · ·+C(An)) = 0 if C(Ak) = 0 (k ∈ N), and then C(S∞ k=1 Ak) = limn→∞ C(A1 ∪ · · · ∪ An) = 0. If χA = χB C-q.e., then C(A) = C(B) by the Fatou property, since fn := χA → χB C-q.e. and C(A) ≤ C(B). Similarly, C(B) ≤ C(A). We consider equivalent two functions, f and g, if they are equal C-q.e. In this case R |f| dC = R |g| dC, since C{|f| > t} = C{|g| > t} for every t ≥ 0. Thus, R |f| dC = 0 if and only if f = 0 C-q.e.

Note that if a Fatou capacity is subadditive, then it is σ-subadditive. The Fatou property can be presented in several equivalent ways:

Theorem 1. The following properties are equivalent:

(a) C *is a Fatou capacity.*
(b) $|f| \le \liminf_n |f_n| \implies f^\star_C \le \liminf_n (f_n)^\star_C$.
(c) $\int (\liminf_n |f_n|)\, dC \le \liminf_n \int |f_n|\, dC$.
(d) $0 \le f_n \uparrow f \implies (f_n)^\star_C \uparrow f^\star_C$.

Proof. (c) follows from (b) and (3), and (a) follows from (c) by taking fn = χAn .

Suppose now that C satisfies (a) and that

$|f| \le \liminf_n |f_n|$. Let $A^t :=$

{|f| > t} and At n := {|fn| > t}. Then,

$$A^t \subset \liminf_n A^t_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A^t_n$$

and, by (a),

$$C(A^t) \le \lim_m C\Big(\bigcap_{n=m}^{\infty} A^t_n\Big) \le \liminf_n C(A^t_n),$$

so that $\chi_{[0,C(A^t))} \le \liminf_n \chi_{[0,C(A^t_n))}$ and (b) follows:

$$f^\star_C(x) = \int_0^\infty \chi_{[0,C(A^t))}(x)\, dt \le \liminf_n \int_0^\infty \chi_{[0,C(A^t_n))}(x)\, dt = \liminf_n (f_n)^\star_C(x).$$

Theorem 2. If 1 ≤ p ≤ ∞ and p 0 = p/(p − 1), then the following versions of H¨older and Minkowski inequalities hold

$$\int_\Omega |fg| dC \le 2c\Big(\int_\Omega |f|^p dC\Big)^{1/p}\Big(\int_\Omega |g|^{p'} dC\Big)^{1/p'}$$

and

$$\Big(\int_\Omega |f+g|^p dC\Big)^{1/p} \le 4c^2\Big[\Big(\int_\Omega |f|^p dC\Big)^{1/p} + \Big(\int_\Omega |g|^p dC\Big)^{1/p}\Big].$$

If the Choquet integral is subadditive (cf. Section 4), then the H¨older and the Minkowski inequalities are satisfied with constant 1:

$$\int_\Omega |fg| dC \le \Big(\int_\Omega |f|^p dC\Big)^{1/p}\Big(\int_\Omega |g|^{p'} dC\Big)^{1/p'}$$

$$\Big(\int_\Omega |f+g|^p dC\Big)^{1/p} \le \Big(\int_\Omega |f|^p dC\Big)^{1/p} + \Big(\int_\Omega |g|^p dC\Big)^{1/p}$$

Proof. We write $|fg| = (|f|^p)^{1/p}(|g|^{p'})^{1/p'}$. Since the inequality $a \le b^\theta c^{1-\theta}$ holds if and only if $a \le \theta\epsilon^{\theta-1} b + (1-\theta)\epsilon^\theta c$ for all $\epsilon > 0$, by taking $\theta = 1/p$, $a = |fg|$, $b = |f|^p$, $c = |g|^{p'}$ we obtain $|fg| \le \theta\epsilon^{\theta-1}|f|^p + (1-\theta)\epsilon^\theta |g|^{p'}$.

Hence, in the subadditive case (in the general case the proof is the same but the constant $2c$ from (4) has to be included), we have

$$\int_\Omega |fg|\, dC \le \theta\epsilon^{\theta-1}\int_\Omega |f|^p\, dC + (1-\theta)\epsilon^\theta \int_\Omega |g|^{p'}\, dC.$$

Denote $A = \int_\Omega |f|^p\, dC$, $B = \int_\Omega |g|^{p'}\, dC$ and $\gamma(\epsilon) = \theta\epsilon^{\theta-1}A + (1-\theta)\epsilon^\theta B$ we have that $\int_\Omega |fg|\, dC \le \gamma(\epsilon)$, and $\gamma(\epsilon) \ge \gamma(\epsilon_0)$ with $\epsilon_0 = A/B$. Hence

$$\int_\Omega |fg|\, dC \le \gamma(\epsilon_0) = \frac{A^\theta}{B^{\theta-1}} = \Big(\int_\Omega |f|^p\, dC\Big)^{1/p}\Big(\int_\Omega |g|^{p'}\, dC\Big)^{1/p'}.$$

The Minkowski inequality (6) follows from (5) in the usual way.

One could wonder if these estimates are always true with constant 1. We will see in Section 4 that sub additivity holds only if C is concave. It is easily checked that H¨older's inequality is always true for sets, since

$$\int \chi_A\chi_B\, dC = C(A\cap B) \le C(A)^{1/p} C(B)^{1/p'},$$

but the following example shows that it is no longer true for functions:

Example 1. Consider the "Lorentz-type" capacity C(A) := R |A| 0 w(t) dt on (0, 1) with w(t) = tχ(0,1)(t), and the functions

$$f(x) = x^{-1/2}, \quad g(x) = x^{1/2} \qquad (on\ (0,1)).$$

Then

$$\int f^2\, dC \int g^2\, dC < \Big(\int fg\, dC\Big)^2.$$

Just note that $\int fg\, dC = C((0,1)) = 1/2$, $\int f^2\, dC = \int_0^1 f(x)^2 x\, dx = 1$ and $\int g^2\, dC = \int_0^1 (1-x)^2 x\, dx = 1/6$.

Hence, there is no hope to obtain the H¨older and Minkowski inequalities with steady 1 in the general case. We don't know whether the subadditivity of the Choquet basic is an important condition to get H¨older's gauge with steady 1

3. Lebesgue capacitary space

Starting now and into the foreseeable future, C will speak to a Fatou limit on (Ω, Σ) and c ≥ 1 its subadditivity steady. In this area we study the fulfillment of the spaces L p,q(C) (p, q > 0) characterized by the condition

$$\|f\|_{L^{p,q}(C)} := \left(q\int_0^\infty t^{q-1}C\{|f|>t\}^{q/p}dt\right)^{1/q} < \infty$$

if $q < \infty$. If $q = \infty$, $\|f\|_{L^{p,\infty}(C)} := \sup_{t>0} tC\{|f|>t\}^{1/p}$.

Observe that $\|f\|_{L^{p,q}(C)} = 0$ if and only if $f = 0$ C-q.e. and equivalent functions (in the sense of C-q.e. identity) have the same $\|\cdot\|_{L^{p,q}(C)}$-norm. Moreover $\|\lambda f\|_{L^{p,q}(C)} = |\lambda|\|f\|_{L^{p,q}(C)}$ and $\|f+g\|_{L^{p,q}(C)} \le 2c(\|f\|_{L^{p,q}(C)} + \|g\|_{L^{p,q}(C)})$.

We write L p (C)

$(C) = L^{p,p}(C)$ if $p < \infty$ with $\|f\|_{L^p(C)} = \left(\int_\Omega |f|^p dC\right)^{1/p}$.

$$\|f\|_\infty := \inf\{M > 0;\ |f| \le M\ C\text{-q.e.}\} < \infty.$$

As for function spaces, there are several descriptions of these "norms"

Theorem 3. $\|f\|_{L^p(C)} = \|f_C^\star\|_p = \||f|^p\|^{1/p} = \left(p\int_0^\infty t^{p-1}C\{|f|>t\}\,dt\right)^{1/p}$.

Proof. Let $\psi(t) = t^p$. Then $\int_0^\infty \psi(f_C^\star(t))\,dt = \int_0^\infty \psi(|f|)_C^\star(t)\,dt$ and, if we denote $g = \psi(|f|)$, an application of Fubini theorem gives

$$\int_0^\infty g_C^\star(t)\,dt = \int_0^\infty\int_0^\infty \chi_{[0,C\{g>x\})}(t)\,dx\,dt = \int_0^\infty\int_0^\infty \chi_{[0,C\{g>x\})}(t)\,dt\,dx = \int_0^\infty C\{g>x\}\,dx,$$

this is, $\int_0^\infty \psi(f)_C^\star(t)\,dt = \int_0^\infty C\{\psi(|f|)>t\}\,dt$.

Also, if $x = \psi(t)$, then

$$\int_0^\infty C\{|f|>t\}\,d\psi(t) = \int_0^\infty C\{|f|>\psi^{-1}(x)\}\,dx = \int_0^\infty C\{\psi(|f|)>x\}\,dx$$

and $\int_0^\infty C\{|f|>t\}\,d\psi(t) = \int_0^\infty C\{\psi(|f|)>t\}\,dt$. □

Theorem 4. *$\|\cdot\| := \|\cdot\|_{L^p(C)}$ is quasi-subadditive, with constant $c_p = (2c)^{1/p}$ if $1 \le p < \infty$ and $c_p = c^{1/p}2^{(2-p)/p}$ if $0 < p < 1$.*

Proof. Suppose $1 \le p < \infty$. By (2),

$$\|f+g\|^p \le \int_0^\infty \left(f_C^\star\left(\frac{x}{2c}\right) + g_C^\star\left(\frac{x}{2c}\right)\right)^p dx = 2c\|f_C^\star + g_C^\star\|_p^p$$

and the results follow from the estimates for $L^p(\mathbf{R}^+)$.

If $p < 1$, then, since $a^p + b^p \le 2^{1-p}(a+b)^p$ $(a, b \ge 0)$, we conclude that

$$\|f_C^\star + g_C^\star\|_p^p \le \int_0^\infty f_C^\star(y)^p\,dy + \int_0^\infty g_C^\star(y)^p\,dy \le 2^{1-p}(\|f_C^\star\|_p + \|g_C^\star\|_p)^p,$$

and $\|f+g\| \le (2c)^{1/p}2^{(1-p)/p}(\|f\| + \|g\|)$. □

Now, recall that if $\|\cdot\|$ is a quasi-seminorm with constant $c \ge 1$ and $(2c)^\varrho = 2$ then, by Aoki's theorem (cf. Section 3.10 of [8]),

$$\|f\|^* := \inf\left\{\sum_{j=1}^n \|f_j\|^\varrho : n \ge 1, \sum_{j=1}^n f_j = f\right\},$$

is a 1-norm $\|\cdot\|^*$ such that

$$\|f\|^* \le \|f\|^\varrho \le 2\|f\|^*,$$

and it follows that

$$\left\|\sum_i |f_i|\right\| \le 2^{1/\varrho}\left(\sum_i \|f_i\|^*\right)^{1/\varrho} \le 2^{1/\varrho}\left(\sum_i \|f_i\|^\varrho\right)^{1/\varrho}.$$

In the special case $f_i = \chi_{A_i}$ and $p = 1$ we obtain

$$C\left(\bigcup_{i=1}^\infty A_i\right)^\varrho \le 2\sum_{i=1}^\infty C(A_i)^\varrho.$$

We say that $\{f_n\}$ converges to f in capacity if $C\{|f_n - f| > \epsilon\} \to 0$ $n \to \infty$, for every $\epsilon > 0$.

Note that if the sequence {fn} converges in capacity, then it is a Cauchy sequence in capacity, that is, for every □ > 0, C{|fp − fq| > □} → 0 as p, q → ∞. The converse is also true:

Theorem A sequence {fn} is convergent in capacity to a function f if and only if it is a Cauchy sequence in capacity. In this case, the sequence has a subsequence which is C-q.e. convergent to f.

Proof. If {fn} is a Cauchy sequence in capacity, then there exists nk ∈ N so that

$$C\{|f_p - f_q| > 2^{-k}\} < 2^{-k} \qquad (p, q \ge n_k > n_{k-1}).$$

Denote $A_k := \{|f_{n_k} - f_{n_{k+1}}| > 1/2^k\}$ and $F_m := \bigcup_{k\ge m} A_k$. If $j \ge i \ge m$, then $|f_{n_i} - f_{n_j}| \le 1/2^{m-1}$ on $\Omega \setminus F_m$. So $\{f_{n_k}\}$ is uniformly Cauchy on $\Omega \setminus F_m$ and it is simply convergent to a function f on $E := \bigcup_{m=1}^\infty (\Omega \setminus F_m)$. By (8) and the Fatou property,

$$\begin{aligned} C(\Omega \setminus E) &\le \lim_{m\to\infty} C(F_m) = \lim_{m\to\infty} \|\chi_{F_m}\|_{L^1(C)} = \lim_{m\to\infty} \|\chi_{\bigcup_{k\ge m} A_k}\|_{L^1(C)} \\ &\le \lim_{m\to\infty} \|\sum_{k\ge m} \chi_{A_k}\|_{L^1(C)} \le \lim_{m\to\infty} 2^{1/\varrho}(\sum_{k\ge m} \|\chi_{A_k}\|_{L^1(C)}^\varrho)^{1/\varrho} \\ &= \lim_{m\to\infty} 2^{1/\varrho}(\sum_{k\ge m} C(A_k)^\varrho)^{1/\varrho} = 0. \end{aligned}$$

By the Fatou property

$$C\{|f_{n_k} - f| > \eta\} = C\{\lim_{j\to\infty} |f_{n_k} - f_{n_j}| > \eta\} \le \lim_{j\to\infty} C\{|f_{n_k} - f_{n_j}| > \eta\} < \epsilon.$$

Since {fn} is a Cauchy sequence in capacity which has a subsequence which is convergent in capacity to f, {fn} converges also to f in capacity.

The topology and the uniform structure of L p (C) are given by the metric d(f, g) := kf − gk ∗ , where k · k∗ is associated to k · kLp(C) as in (7).

Theorem 6. *$L^p(C)$ $(0 < p < \infty)$ is complete.*

Proof. We follow some usual arguments of measure theory combined with (9):

Let $\{f_n\} \subset L^p(C)$ be a Cauchy sequence. For each k ∈ N, pick nk > nk−1 so that

$$\|f_m - f_n\|^p = \int |f_m - f_n|^p\,dC < \frac{1}{3^k} \qquad (m, n \ge n_k).$$

If $A_k = \{|f_{n_{k+1}} - f_{n_k}|^p > 1/2^k\}$, then $C(A_k) < 2^k/3^k$ since

$$\frac{C(A_k)}{2^k} \le \int_{A_k} |f_{n_{k+1}} - f_{n_k}|^p\,dC < \frac{1}{3^k}.$$

Note that

$$\sum_{k=1}^\infty |f_{n_{k+1}}(t) - f_{n_k}(t)| < \infty \quad \forall t \notin \bigcup_{k>N} A_k$$

because $|f_{n_{k+1}}(t) - f_{n_k}(t)| \le 1/2^{k/p}$ if $k > N$. Therefore, there exists

$$f(t) := f_{n_1}(t) + \sum_{k=1}^\infty (f_{n_{k+1}}(t) - f_{n_k}(t)) = \lim_k f_{n_k}(t) \quad \forall t \notin A = \bigcap_{N=1}^\infty \bigcup_{k>N} A_k$$

and $C(A) = 0$ since, by (9),

$$C(A)^\varrho \le C\Big(\bigcup_{k>N} A_k\Big)^\varrho \le 2\sum_{k>N}\left(\frac{2}{3}\right)^{\varrho k}$$

and $\sum_{k>N}(2/3)^{\varrho k} < \infty$. Put $f(t) := 0$ if $t \in A$.

As $n_k \to \infty$, $|f_{n_k}(t) - f_n(t)|^p \to |f(t) - f_n(t)|^p$ C-q.e. and

$$\int |f - f_n|^p\,dC \le \liminf_k \int |f_{n_k} - f_n|^p\,dC \le \varepsilon$$

for n huge enough. The evidence of culmination of L p (C) can be effectively adjusted to demonstrate that all L p,q(C)- spaces are additionally finished.

Comment 1. The nonappearance of additivity for the Choquet necessary makes it hard to give a portrayal of the double of L p (C). See for example [1, Section 4], where duality on account of Hausdorff and Bessel limits is considered. In the event that p 0 is the conjugate example of p ∈ [1,∞], H¨older's imbalance demonstrates that each g ∈ L p 0 (C) + characterizes a useful ug(f) := R fg dC which is homogeneous and limited on L p (C)+

$$u_g(f) \leq 2c\Big(\int g^{p'}\,dC\Big)^{1/p'}\Big(\int f^p\,dC\Big)^{1/p},$$

but in general ug is not additive. The Choquet integral is sub additive on set

$$\int(\chi_A+\chi_B)\,dC \leq \int \chi_A\,dC + \int \chi_B\,dC,$$

if and only if

$$C(A\cup B)+C(A\cap B)\leq C(A)+C(B).$$

At that point the Choquet necessary is likewise sub added substance on nonnegative basic capacities. These actualities were demonstrated by Choquet in [16] (see likewise [15] or [14] for a direct rudimentary confirmation). For this situation C is said to be firmly subadditive or inward. Variety limits and those of Fuglede and Meyers are instances of sunken limits. Shannon entropy is inward if n = 1, yet not if n > 1 (see [17]). On account of the entropies CE related to Banach capacity spaces, models and counterexamples of sunken limits are surrendered

Sunken limits offer ascent to normed L p - spaces, since the Minkowski disparity holds with steady 1, and a characteristic inquiry is to decide when, for a non-inward limit C, L p (C) is standard capable, this implying there exists in L p (C) a standard which is equal to k • kLp(C) .

With respect to common Lorentz spaces, one could attempt to substitute f ? C by

$$f^{\star\star}(t)=\frac{1}{t}\int_0^t f_C^\star(s)\,ds,$$

but unfortunately this average function is subadditive precisely when L p (C) (p ≥ 1) are normed spaces:

Theorem . f ?? is subadditive with respect to f if and only if C is concave.

Proof. It is clear that Ct(A) := min(C(A), t) is a Fatou capacity. For a fixed t > 0, f ??(t) is subadditive in f if and only if Ct is concave, since

$$\int_0^t f_C^\star(s)\,ds=\int_0^\infty dy\int_0^t \chi_{[0,C\{f>y\})}(s)\,ds=\int_0^\infty \min(C\{f>y\},t)\,dy,$$

what's more, the hypothesis pursues. We don't have an acceptable adequate normability condition. Give us a chance to see a prohibitive one, which broadens a known outcome for old style Lorentz spaces. In the remainder of the area µ speaks to a measure on (Ω, Σ) with the end goal that µ(σ) = [0, µ(ω)] ⊂ [0,∞], and we will guess that C is µ-invariant, this implying C(A) = C(B) if µ(A) = µ(B). A limit C on (Ω, Σ) will be said to be semi curved regarding µ if there exists a steady γ ≥ 1 with the end goal that, at whatever point µ(A) ≤ µ(B), the accompanying two conditions are fulfilled:

- $C(A)\leq\gamma C(B)$, and
- $\frac{C(B)}{\mu(B)}\leq\gamma\frac{C(A)}{\mu(A)}$,

that is, for all $A,B\in\Sigma$,

$$C(B)\leq\gamma\max\Big(1,\frac{\mu(B)}{\mu(A)}\Big)C(A).$$

Example 2. If J : [0, µ(Ω)] → R is an increasing function such that J(t)/t is decreasing, then it is readily seen that C(A) := J(µ(A)) defines a µ-invariant and quasi-concave capacity with respect to µ. For instance, C(A) := ϕX(µ(A)) when ϕX is the fundamental function of an r.i. space. Note that ϕX is a quasi-concave function.

Theorem 8. If the capacity C is µ-invariant and quasi-concave with respect to µ, then

$$\tilde{C}(A):=\sup\Big\{\sum_{i=1}^n\lambda_iC(A_i):\,n\in\mathbf{N},\,\sum_{i=1}^n\lambda_i=1,\,\lambda_i\geq0,\,\sum_{i=1}^n\lambda_i\mu(A_i)\leq\mu(A)\Big\}$$

defines a concave capacity and

$$\bar{C}(A):=\inf_{A_n\uparrow A,\,A_n\in\Sigma}\Big\{\lim_{n\to\infty}\tilde{C}(A_n)\Big\}$$

A concave Fatou capacity. Both Ce and C¯ are equivalent to C.

Proof. It is clear that C˜(A) ≥ 0 and it is readily seen that C˜ is increasing. Let us show that

$$C(A)\leq\tilde{C}(A)\leq2\gamma C(A)$$

Obviously C(A) ≤ C˜ P (A). On the other hand, if ε > 0 is given, we can find n i=1 λiμ(Ai) ≤ μ(A) with Pn i=1 λi = 1 and λi ≥ 0 such that

$$\tilde{C}(A) - \varepsilon \leq \sum_{i=1}^{n} \lambda_i C(A_i) \leq \gamma \sum_{i=1}^{n} \lambda_i \max\left(1, \frac{\mu(A_i)}{\mu(A)}\right) C(A) \leq 2\gamma C(A)$$

and (10) follows

To prove that $\tilde{C}$ is concave, let $0 < \theta < 1$ and $\varepsilon > 0$. If $A, B \in \Sigma$, we can find $\sum_{i=1}^{n} \lambda_i \mu(A_i) \leq \mu(A)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ such that

$$(1-\theta)\widetilde{C}(A) - \frac{\varepsilon}{2} \leq (1-\theta) \sum_{i=1}^{n} \lambda_i C(A_i)$$

and, similarly,

$$\theta\widetilde{C}(B) - \frac{\varepsilon}{2} \leq \theta \sum_{j=1}^{m} \lambda'_j C(B_j).$$

with $\sum_{j=1}^{m} \lambda'_j \mu(B_j) \leq \mu(B)$ with $\sum_{j=1}^{m} \lambda'_j = 1$ and $\lambda'_i \geq 0$.

Then $(1-\theta)\mu(A) + \theta\mu(B) \geq \sum_{i=1}^{n} (1-\theta)\lambda_i \mu(A_i) + \sum_{j=1}^{m} \theta\lambda'_j \mu(B_j)$ and $\sum_{i=1}^{n} (1-\theta)\lambda_i + \sum_{j=1}^{m} \theta\lambda'_j = 1$. We can choose $D \in \Sigma$ such that $\mu(D) = (1-\theta)\mu(A) + \theta\mu(B)$, and then

$$(1-\theta)\widetilde{C}(A) + \theta\widetilde{C}(B) - \varepsilon \leq \sum_{i=1}^{n} (1-\theta)\lambda_i C(A_i) + \sum_{j=1}^{m} \theta\lambda'_j C(B_j) \leq \widetilde{C}(D),$$

so that

$$(1-\theta)\widetilde{C}(A) + \theta\widetilde{C}(B) \leq \widetilde{C}(D). \qquad (11)$$

Since C is μ-invariant, the same happens with $\widetilde{C}$ and we may define $\varphi(s) := \widetilde{C}(A)$ if $s = \mu(A)$, which is a concave function on $[0, \mu(\Omega)]$, by (11).

We claim that, if $x, y \geq t > 0$, then

$$\varphi(x+y-t) + \varphi(t) \leq \varphi(x) + \varphi(y), \qquad (12)$$

and the concavity of $\tilde{C}$ follows by taking $t = \mu(A \cap B)$, $x = \mu(A)$ and $y = \mu(B)$, since then $\varphi(t) = \widetilde{C}(A \cap B)$, $\varphi(x+y-t) = \varphi(\mu(A) + \mu(B) - \mu(A \cap B)) = \varphi(\mu(A \cup B)) = \widetilde{C}(A \cup B)$, and $\varphi(x) + \varphi(y) = \widetilde{C}(A) + \widetilde{C}(B)$.

To prove the claim, we may assume that $0 < t < x \leq y$ and write

$$x = (1-\tau)t + \tau(x+y-t), \quad y = (1-\tau')t + \tau'(x+y-t) \qquad (\tau, \tau' \in (0,1)).$$

Let X be a compact Hausdorff space and let C(X) (CR(X)) denote the set of all complex-valued (real-valued) continuous functions on X. With usual operations of addition, multiplication and scalar multiplication and with the norm defined by

$$||f|| = \sup \{ |f(x)| : x \in X \}$$

for f e CCX) CCR(X)), CCX) (CRCX)) is a complex (real) Banach algebra with identity. A function algebra on X is a closed subalgebra of CCX) which contains constants and separates the points of X.

A decomposition of X is a collection of disjoint closed subsets of X whose union is X. Subalgebras of C(X) (CR(X)) and the decompositions of X are closely related. For example, a closed ideal of C(X) is determined by a closed subset of X. Now, if F is a closed subset of X, then we can associate with it the decomposition $\mathcal{D} = \{F\} \cup \{\{x\} : x \in X-F\}$. Thus every closed ideal is associated with a decomposition of X consisting of a closed set and singletons outside the closed set. If A is a closed subalgebra of CR(X) (a self-conjugate closed subalgebra of CCX» containing constants, then the sets of constancy of A gives a decomposition which is upper semicontinuous. Conversely, if 5) is an upper semicontinuous decomposition of X, then there exists a unique closed subalgebra of CR(X) containing constants whose sets of constancy are precisely the members of 2). This association of decompositions of X and subalgebras of CR(X) has been found very useful in the study of CR(X) as a direct sum of two subalgebras.

The role of decompositions in the study of function algebras was highlighted by Silov and more so by Bishop. The Silov decomposition for a function algebra A on X consists of sets of constancy of $A_R = A \cap C_R(X)$. The Bishop decomposition for A consists of maximal sets of antisymmetry. Both these decompositions have the following crucial property *

If $f \in C(X)$ and $f|_E \in (A|_E)^-$ for every member E in the decomposition, then f e A. The above property is known as the (D)-property in the literature.

Once the importance of decompositions is recognised, it is natural to ask further questions. Some of the questions are

(1) Are there decompositions, other than Silov and Bishop, associated with a function algebra which also have the (D)-property?

(2) Does a Bishop (Silov) decomposition have a stronger property than the (D)-property?

(3) How are Bishop, Silov and other decompositions related to each other? Do some of these decompositions determine the others?

(4) Does every member of a decomposition satisfying property such as (D)-property have any special property in relation to a function algebra? (For example, every member of Bishop decomposition of a function algebra is an intersection of peak sets),

(5) How are the decompositions of A and ft related, where it is the algebra of Gelfand transforms of A?

(6) Can the decompositions analogous to Silov and Bishop for a function algebra be defined for a function space? What are their properties?

(7) How about the decompositions for a real function algebra ? for an algebra of vector-valued continuous functions? Some of these and related questions have been discussed in the literature. This thesis deals with further investigations of these questions.

Before we give the chapter wise summary of the results proved in the thesis, it will be convenient to set up notations and give definitions and other preliminaries.

1. Preliminaries

Let X be a compact Hausdorff space. A decomposition of X is $_a$ collection of disjoint closed subsets of X whose union is X. We shall denote decompositions by ξ, S', y $^{etc. First}$ we $^{define\ certain}$ notions related to decomposition of X.

Definitions 0.1.1 [20, p.4]. (i) Let ξ_1 and ξ_2 be two decompositions of X. If for every $E_1 \in \xi_1$, there exists $E_1 \subset E_2$ such that $E_1 \subset E_2$ then ξ_1 is said to be finer than ξ_2 and we write $\xi_1 < \xi_2$

It is clear that if ξ< ξ and ξ < ξ, then $_{1\ 2\ 2\ 1}$ ξ = ξ . If ξ < 8 and 8*8 , then we write 8^8 .

(ii) Let 8 be a decomposition of X and F be a closed subset of X. Then 8nF={EnFs E n F * <£, E « 8 }.

(iii) A set F is said to be saturated with the decomposition 8 of X if, whenever E e 8 and E n F * <p, then E c F.

Definition 0.1.2. A decomposition S of X is said to be upper semicontinuous (u.s.c.) if for each E e 8 and each open set U containing E, there is an open set V such that E c V c U and if E n V * <p for E e «, then E c U.

Note that if $ is an u.s.c. decomposition of X, then the quotient space X/8 is Hausdorff. Now, we define some well-known ideas related to a function algebra. For details, we refer to. Let A denote a function algebra on X, i.e., A is a closed subalgebra of CCX) which contains constants and separates the points of X. The maximal ideal space m(A) of A is the set of all nonzero complex homomorphisms on X. % - { t i f e A } denotes the Gelfand transform of A. For a /V subset E of X, the A-hull of E is the set E = { 0 e m(A) •| f (0) | ||f||E for all f e A }, where ||f||E= sup { |f(x)| : x e E). A probability measure p on X is said to be a representing measure for 0 e m(A) with respect to A if J fd£i = 0(f) for every f in A. The essential set of A is the X$_{hull\ of\ the\ largest\ closed\ ideal\ of\ C(X)\ contained\ in\ A.\ It}$ is denoted by ECA}. If ECA} = X, then A is called an essential algebra.

Examples 0.1.3. Let X denote a compact subset of the complex plane €,

(i) Define ACX) = { f « CCX> * f is analytic in the interior of X V. Then ACX) is a function algebra on X. If X = D = { z e C : | z| < 1), then A(D) is called the disk algebra on the unit disk D. The restriction of ACD) to the unit circle T is called the disk algebra on the unit circle T.

(ii) Let PCX) denote the uniform closure of all polynomials in z. Then PCX) is a function algebra on X and PCX} c ACX).If A and B are function algebras on compact Hausdorff spaces X and Y respectively, then we can construct function algebras on X x Y, naturally associated with A and B namely the tensor product A ® B and the slice product A # B of A and B, which are defined as follows.

For f e A and g e B, define f ® g on X x Y by Cf ® g)Cx,y) = fCx)gCy). Then f ® g e OCX x Y). The space of all finite linear combinations of functions of the type f ® g, f e A, g e B is called the algebraic tensor product of A and B and is denoted by A ® B. In fact, A ® B is a subalgebra of CCX x Y) which separates the points of X x Y and contains the constant functions. The uniform closure of A ® B in C(X x Y) is called the tensor product of the function algebras A and B and is denoted by A ® B.

For function algebras A and B on X and Y, take A # B = \ f e C(X x Y) : f e A, fx e B for all x e X and y e Y where for a fixed y e Y, f Cx) = f(x,y) for each x e X and for a fixed x e X, f $_X$ (y) = f(x,y) for each ye Y. It can be easily verified that A # B is a closed subalgebra of CCX x Y> and A & B c A # B. Hence A # B is a function algebra on X x Y. It is called the slice product of A and B.

We shall have several occasions to use the lemma given below in the chapters that follow.Lemma 0.1.4. Let A and B be function algebras on X and Y respectively. Let E and F be closed subsets of X and Y. Then

(i) (a # b|ExF] «ca|ej # cb|f).

Proof. (i) It is clear that A ® B|ExF = A|E ® B|F . Let f e A B|gxj, . Then f = g|gxp for some g e A S B. Therefore, There exists a sequence { g $_n$ i in A ® B such that Hence f e CA|£> $ CB| p . Thus A $ B|ExF e CAf E> § CB| p . Since the latter is closed, CA $ blexF^ c ^ale^ ® CB|FJ. Conversely, let f e (A|E> ® CB|F). Then f = £ g. » h. , where L = i. gt e (A|g3 and h <s CBjjP for i = 1,2, ... ,n. Thus for each i, there exist sequences { 0ik 1 in A and { 1 in B such that <p.$_{ik}$ gi and J $_{1F}$ —* h uniformly, as k —> od. Therefore, for each i, 4>.,® $_{tk}$ W., $_{lk«ExF}$ I = 0., $_{lk'E}$ I ® $_{tk'F}$ I —> g. ® h uniformly. Thus $_{i.}$ £ $_{=\ i}$ C0ik $_1$ ® Wifc5l $_{x\ i\ ExF}$ £g.» ▪▪$_t$h. $_i$ = f But v =i t i. = 1 ' JlXiJ « A ® blexf = A 8 ®l ^{ExF} Hence f e (A S B|ExF> and (Ale^ ® c ^A ^ BlexF^' Consequently, CAjgJ & (B|jJ <= CA § blexF^'

(ii) Let f e A # B|ExF . Then there is g e A # B such that f = ®lexF ' x e Then gx e B and g I e B|F . Thus ' F fx « blf = cb|f>. This is true for each x e E. Similarly, for each y e F, f e CA|F). Hence f e CA|F) # (B|F) and A #

B|ExF c CA|e) # CB|fJ. Thus we CA # B|ExF> c CA|g) # CB|f>.

get We do not know whether CA # B|ExF) = ^Ale^ # is true or not.

Our study is mainly concentrated on the Bishop and Silov vdecompositions. For function algebras, these decompositions appear in literature at many places. See, for example, and.

Definitions (i) A subset K of X is said to be a set of antisymmetry or an antisymmetric set for a function algebra A if whenever f e A and f|^ is real-valued, then f|is constant. The collection of all maximal sets of antisymmetry for A forms a decomposition of X. It is called the Bishop decomposition for A and is denoted by 9C(A).

(ii) A set of constancy of Ar is called a Silov set for A, where A = A n C (X). R R

The collection of all maximal Silov sets for A is v clearly a decomposition of X, called the Silov decomposition for A. We denote it by ^(A).

It is clear from the definitions that 9CCA) < nA). Next, we define certain ideas for a subspace of C(X}.

Definition. A closed subspace A of CCX5 which contains constants is called a function space on X.

Now onwards, A denotes a function space on X.

A closed subset E of X is called a closed restriction set (CR set) for A if A|g is closed in 0(E). E is called an interpolation set for A if Ajg = C(E). Let MOO denote the set of all regular Borel measures on X. Then the annihilator of A is J fd£< = 0 for X

Definitions Let A be a function space on X and F be a closed subset of X.

(i) F is called a peak set for A if there exists f e A such that f|j, = 1 and | f(x) | < 1 for every x e X-F. The intersection of peak sets is called a generalized peak set for A.

(ii) F is called a p-set for A, if ^ s A1 4 e h~, where ^F(G) = jj(F n G) for every Borel subset G of X.

Remarks (i) If F is a p-set for A, then F is a CR set for A.

(ii) It is proved in [13, Proposition 1.5] that a p-set for function space A is a generalized peak set for A. But a generalized peak set may not be a p-set for a function space.

(iii) If A is a function algebra on X, then F is a p-set for A if and only if F is a generalized peak set for A.

Remark 0.1.9. If A and B are function spaces on X and Y respectively, then we can define A $ B and A # B exactly as we have defined for function algebras. Also, it can be checked that Lemma 0.1.4 remains true in this case.

Finally, we define some properties of a decomposition of X which are associated with A.

Definitions. Let A be a function space on X and * be a decomposition of X.

(i) We say that * has the CD)-property for A if f e CCX) and f|g e CA|gJ for every Eg* implies that f e A, where CA|gJ denotes the uniform closure of A|g in CCE).

(ii) We say that * has the C S) -property for A if, whenever F is a p-set for A and is saturated with *, then * n F has the CD)-property for A|j. .

(iii) We say that * has the CGA)-property for A if, whenever |i| A A p e bCA) , then supp /j c E for some E e 8, where bCA) denotes the set of extreme points of the unit ball bCA^) of a\

Remarks

(i) For a decomposition * of X, we have CGA)~property CS)-property *> CD)-property.

(ii) If and "&2 are two decompositions of X such that 8 < «2 and if 8^ has any one of the above properties for A, then «2 also has the same property for A.

Finally, we define real function algebra and a vector function space on X, since we shall be discussing decompositions for them.

Let X be a compact Hausdorff space and t : X -—» X be a homeomorphism on X such that tot is the identity map on X. Then CCX.t) = i f e CCX) s f(x) = f(rCx)) for all x e X } is a commutative real Banach algebra with identity.

Definition -A subalgebra A of CCX,t) is called a real function algebra on (X,t) if

(i) A is uniformly closed;

(ii) A contains Creal) constants and

(iii) A separates the points of X.

Let X be a compact Hausdorff space and B be a commutative Banach algebra with identity. Let CCX;B) denote the algebra of continuous, B-valued functions on X. Then CCX;B) is a commutative Banach algebra with identity under pointwise operations and the norm given by

1 | f | | = sup { 1 |fCaOj |B s xe X J , f e CCX; B>.

Definition- 0.1.13. (i) A vector function space on X is a closed subspace of CCX;B) which contains vector constants.

(ii) A vector function algebra on X is a closed subalgebra of CCX;B) which contains vector constants and separates the points of X.

CONCLUSION

Different detachment sayings are examined in Topology. When we manage a function variable based math on a smaller Hausdor space X, the space X has all the decent partition properties for the most part because of the Urysohn's lemma. These topological properties all being moved to function algebras and all the more for the most part to commutative Banach algebras. The consistency and typicality properties were considered a lot before. Presently if A will be a function variable based math on X, it can likewise be acknowledged as a function polynomial math on (A). So these properties can be de ned on X or on (A). It is intriguing to take note of that typicality suggests consistency yet normality does not infer ordinariness all in all However, on (A), both these ideas correspond. The Cartesian result of normal commutative Banach algebras with personality is contemplated in.

REFERENCES

1. Glicksberg (1962). Measures orthogonal to algebras and sets of antisymmetry, Trans. Amer. Math. Soc. 105, pp. 415-435.

2. H.Kulkarni and B.V. Limaye (1981). Gleason parts of real function algebras, Can. J. Math. 33, pp. 181-200.

3. H. Kulkarni and N. Srinivasan (1988). An analogue of Hoffman-Wermer theorem for a real function algebra, Indian J. Pure appl. Math. 19(2).

4. H.S. Mehta, R. D. Mehta and M. H. Vasavada, Silov and other decompositions for a real function algebra, Math. Today (to be published).

5. Hoffman and J. Wermer (1962). Characterization of C(X), Pacific J. Math. 12, pp. 941-944.

6. J.Ellis (1973). Central decompositions and the essential set for the space ACK), Proc. London Math. Soc. 26, pp. 564-576.

7. L. Asimow and A. J. Ellis (1980). Convexity theory and its applications in Functional Analysis, Academic Press, London.

8. L. Asimow (1970). Decomposable compact convex sets and peak sets for function spaces, Proc. Amer. Math. Soc. 25(1), pp. 75-79.

9. S. H. Kulkarni and N. Srinivasan (1987). An analogue of Bishop's theorem for real function algebras, Indian J. Pure appl. Math. 18(2), pp. 136-145.

10. S. Grigoryan and T. Tonev (2006). Shift Invariant Uniform Algebras on Groups, Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series), 68. Birkhauser Verlag, Basel-Boston-Berlin.

11. S. H. Kulkarni and N. Srinivasan: An analogue of Wermer's theorem for a real function algebra.

12. S.J. Sidney (1972). On the Silov and Bishop decompositions of a uniform algebra, Duke Math. J. 39, pp. 551-559.

13. Srinivasan and S. H. Kulkarni (1988). Restriction algebras of real function algebra, J. Math. Phy. Sci. 22, pp. 209-223.

14. T.W. Gamelin (1969). Uniform Algebras, Prentice-Hall, Inc., Englewood Cliffs, N.J.

Corresponding Author

Kusum Devi*

Research Scholar of OPJS University, Churu, Rajasthan