Journal of Advances and Scholarly Researches in Allied Education
Vol. 13, Issue No. 1, April-2017, ISSN 2230-7540

An Analysis on Metric Spaces and Its Continuity: Some Aspects

Versha Chopra*

Abstract – Metric spaces, which generalize the properties of usually experienced physical and abstract spaces into a mathematical structure. This paper will acquaint the peruser with the idea of metric spaces and continuity. A ton accentuation has been given to propel the thoughts under talk to enable the peruser to create ability in utilizing his creative mind to envision the abstract idea of the subject. Assortment of examples alongside genuine applications have been given to comprehend and value the excellence of metric spaces.

---------------------------------X---------------------------------

INTRODUCTION

A metric space is a non-empty set furnished with structure controlled by a well-characterized thought of distance. A large number of the arguments you have found in a few variable calculus are practically indistinguishable from the relating arguments in single variable calculus, particularly arguments concerning convergence and continuity. The explanation is that the thoughts of convergence and continuity can be formulated as far as distance, and that the idea of distance between numbers that you need in single variable theory, is fundamentally the same as the thought of distance between points or vectors that you need in the theory of functions of severable variables. In further developed science, we have to discover the distance between more convoluted articles than numbers and vectors, for example between groupings, sets and functions. These new thoughts of distance prompts new ideas of convergence and continuity, and these again lead to new arguments surprisingly like those you have just found in single and a few variable calculus.

Sooner or later it turns out to be very exhausting to perform nearly similar arguments again and again in new settings, and one starts to think about whether there is general theory that covers every one of these examples {is it conceivable to build up a general theory of distance where we can demonstrate the outcomes we need for the last time? The answer is truly, and the theory is known as the theory of metric spaces.

Metric Space-

Give x a chance to be any set and give $d: X \times X \to \mathbb{R}$ a chance to be a genuine esteemed capacity fulfilling the accompanying properties:

P1. $d(x,y) \geq 0$ for all $x, y \in X;$

P2. $d(x,y) = 0 \Leftrightarrow x = y$

P3. d(x,y) = d(y,x) for each of the $x, y \in X$

P4. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$

The capacity d is known as a metric on x (in some cases the distance work on x). The ordered pair (x,d) is known as a metric space. In this manner a metric space comprises of a non-empty set furnished with an idea of distance (metric). In the event that there is no equivocalness on the metric considered, at that point we essentially mean the metric space (x,d) by x. We allude the components in x as points and d(x,y) as the distance between the points x and y.

Inconsequentially, an empty capacity is the main metric on the empty set. Additionally, inferable from condition second, the main metric on a singleton set is the zero capacity.

METRIC SPACES : SOME EXAMPLES

Example 1 : The Real Line $\mathbb{R}$

Let $\mathbb{R}$ be the set of all real numbers and $u: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined as

$$u(x,y) = |x - y| \quad \forall\ x, y \in \mathbb{R}.$$

Then we shall prove that u is a **metric** on $\mathbb{R}$

First observe that by definition, $(x,y) \geq 0\ \forall\, x, y \in \mathbb{R}$. Therefore **PI** holds.

For any x,y in $\mathbb{R}$, $u(x,y)=0 \Leftrightarrow |x-y|=0 \Leftrightarrow x=y.$

Therefore P2 holds.

Again, for any x,y in $\mathbb{R}$, $u(x,y)=|x-y|=|y-x|=u(y,x).$

Therefore **P3** holds.

To see the triangle inequality **(P4),** suppose $x,y,z \in \mathbb{R}$ be an three points.

Consider

$$\begin{aligned} u(x,y) &= |x-y| \\ &= |(x-z)+(z-y)| \\ &\le |x-z|+|z-y| \\ &= u(x,z)+u(z,y). \end{aligned}$$

It follows that $u(x,y) \le u(x,z)+u(z,y) \quad \forall \ x,y,z \in \mathbb{R}.$

Thus all the four axioms are satisfied. Hence u is a metric on $\mathbb{R}$ and the ordered pair $(\mathbb{R},u)$ is a metric space. The metric u is called the usual or standard metric or Euclidean metric on $\mathbb{R}$.

Example 2 : The Euclidean Metric on $\mathbb{C}$ (Extension of Euclidean metric on $\mathbb{R}$)

Let $\mathbb{C}$ be the set of all complex number and $d:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$ be a function defined as

$$d(z,z') = |z-z'| \quad \forall \ z,z' \in \mathbb{C}\ .$$

Then d is a metric on $\mathbb{C}$, called the **usual metric or Euclidean Metric** on $\mathbb{C}$. Of course, d is an extension to $\mathbb{C}\times\mathbb{C}$ of the Euclidean metric u on $\mathbb{R}$ i.e., $u=d|_{\mathbb{R}}$.

Example 3 : Maximum Metric on $\mathbb{R}^2$

Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers and $d:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ be a function defined as

$$d(x,y)=\max\{|x_1-y_1|,|x_2-y_2|\}$$
$$\forall\ x=(x_1,x_2),\ y=(y_1,y_2)\in\mathbb{R}^2.$$

We shall show that d is a metric on $\mathbb{R}^2$.

By definition, d is a non-negative function and hence **PI** holds.

For **P2,** consider any $x=(x_1,x_2),\ y=(y_1,y_2)\in\mathbb{R}^2,$

$$\begin{aligned} d(x,y)=0 &\Leftrightarrow \max\{|x_1-y_1|,|x_2-y_2|\}=0 \\ &\Leftrightarrow |x_1-y_1|=0 \ \ and \ \ |x_2-y_2|=0 \\ &\Leftrightarrow x_1=y_1 \ \ and \ \ x_2=y_2 \\ &\Leftrightarrow (x_1,x_2)=(y_1,y_2) \ \ i.e., \quad x=y\ . \end{aligned}$$

For any $x=(x_1,x_2), y=(y_1,y_2)\in\mathbb{R}^2,$

$$\begin{aligned} d(x,y) &= \max\{|x_1-y_1|,|x_2-y_2|\} \\ &= \max\{|y_1-x_1|,|y_2-x_2|\} \\ &= d(y,x)\,. \end{aligned}$$

Thus **P3** is satisfied.

To see triangle inequality **(P4),** let

$x=(x_1,x_2), y=(y_1,y_2), z=(z_1,z_2)\in\mathbb{R}^2$ be any points in $\mathbb{R}^3$. Consider

$$\begin{aligned} |x_1-y_1| &= |(x_1-z_1)+(z_1-y_1)| \\ &\le |x_1-z_1|+|z_1-y_1| \end{aligned}$$

$$\le \max\{|x_1-z_1|,|x_2-z_2|\}+\max\{|z_1-y_1|,|z_2-y_2|\}$$
$$= d(x,z)+d(z,y)$$

i.e., $|x_1-y_1|\le d(x,z)+d(z,y)$ ------------------------(A)

Similarly,

$|x_2-y_2|\le d(x,z)+d(z,y)$. ------------------------ (B)

From (A) and (B) it follows that

$$\max\{|x_1-y_1|,|x_2-y_2|\}\le d(x,z)+d(z,y)$$

i.e., $d(x,y)\le d(x,z)+d(z,y)$.

Hence the triangle inequality holds and therefore d is a metric on $\mathbb{R}^2$.

GEOMETRY OF METRIC SPACES

Before we take a gander at what it implies for a grouping to be convergent regarding a given metric, we invest a little energy examining one method for increasing some comprehension about the geometric importance of a given metric.

In the last subsection, we met three distinct metrics: the discrete metric, the taxicab metric on the plane and a blended metric on the plane (which was shaped from the typical distance in R together with the discrete metric).

A simple method to increase some understanding into the conduct of a metric is to take a gander at the balls around a given point. For the standard Euclidean distance in $\mathbb{R}^n$, a bundle of sweep r around a point $a\in\mathbb{R}^n$ comprises of each one of those points whose distance from an is all things considered r, and this definition normally reaches out to general metric spaces. In any case, in the accompanying definition we take care to recognize balls that incorporate points at precisely distance r from the inside an and those that don't.

Definition 1 Open and closed balls

Let (X, d) be a metric space, and let $a \in X$ and $r \geq 0.$

The open ball of radius r with centre a is the set

$$B_d(a,r) = \{x \in X : d(a,x) < r\}.$$

The closed ball of radius r with centre a is the set

$$B_d[a,r] = \{x \in X : d(a,x) \leq r\}.$$

The sphere of radius r with centre a is the set

$$S_d(a,r) = \{x \in X : d(a,x) = r\}.$$

When r = 1, these sets are called respectively the unit open ball with centre a, the unit closed ball with centre a and the unit sphere with centre a.

SEQUENCES IN METRIC SPACES

Since we have a few examples of metric spaces accessible to us, we come back to the problem of characterizing continuous functions between metric spaces.

Since the definition of a general metric space is displayed on the properties of the Euclidean metric $d^{(n)}$ on $\mathbb{R}^n$, and we characterized continuity of functions between Euclidean spaces as far as convergent sequences, it is natural to endeavor to expand our thoughts regarding convergent sequences in $\mathbb{R}^n$ to general metric spaces. Truth be told, we did a great part of the difficult work when we generalized from the thought of convergence for genuine esteemed sequences to that of convergence of sequences in $\mathbb{R}^n$; it is currently just a short advance to build up these ideas for the metric space setting.

We saw that a genuine grouping can be thought of as a function $a : \mathbb{N} \to \mathbb{R}$, given by $n \mapsto a_n$. Note that the only role played by $\mathbb{R}$ here is as the codomain of the function $a : \mathbb{N} \to \mathbb{R}$; the structure of $\mathbb{R}$ becomes important just when convergence is considered. Since the codomain of a function is basically a set, the accompanying definition is a natural generalization.

Definition 1 Sequence in a metric space

Let A be a set. A sequence in X is an unending ordered list of elements of X:

$$a_1, a_2, a_3, \ldots.$$

The element a* is the kth term of the sequence, and the whole sequence is denoted by (a_k), $(a_k)_{k=1}^{\infty}$ or $(a_k)_{k \in \mathbb{N}}$.

Note that this definition of a sequence does not require that we impose any additional structure (such as a metric) on the set X.

The definition of what it means for a sequence to converge in a metric space (X, d) is closely based on the definition of convergence in $\mathbb{R}^n$

Definition 2 Convergence in a metric space

Let (X.d) be a metric space. A sequence (a_k) in X d-converges to $a \in X$ if the sequence of real numbers $(d(a_k, a))$ is a null sequence.

We write $a_k \xrightarrow{d} a$ as $k \to \infty$, or simply $a_k \to a$ if the context is clear.

We say that the sequence (a_k) is convergent in (X,d) with limit a.

A sequence that does not converge (with respect to the metric d) to any point in X is said to be d-divergent..

CONTINUITY IN METRIC SPACES

Now that we know what it means for a sequence to converge in a metric space, we can formulate a definition of continuity for functions between metric spaces.

Definition 1 Continuity for metric spaces

Let (X, d) and (Y, e) be metric spaces and let $f : X \to Y$ be a function.

Then f is (d, e)-continuous at $a \in X$ if:

Whenever (a_k) is a sequence in X for which $a_k \xrightarrow{d} a$ as $k \to \infty$, then the sequence $f(a_k) \xrightarrow{e} f(a)$ as $k \to \infty$.

If f does not satisfy this condition at some $a \in X$-that is, there is a sequence (x_k) in X for which $x_k \to a$ as $k \to \infty$ but $f(x_k)$ does not converge to $f(a)$ then we say that f is (d, e)-discontinuous at a.

A function that is continuous at all points of X is said to be (d, e)-continuous on X (or simply continuous, if no ambiguity is possible).

METRIC SUBSPACES AND METRIC SUPERSPACES

Definition 1 (Subspace of a Metric Space)

Let (x,d) be a metric space and $Y \subseteq X$ be any nonempty set. Let d_Y be the restriction of d to the set Y x Y i.e, $d_Y : Y \times Y \to \mathbb{R}$ given as

$d_Y(x,y) = d(x,y) \;\; \forall\, x,y \in Y$ i.e., $d_Y = d|_{Y \times Y}$.

Since d is a metric on x, along these lines the mapping d_Y is a metric on Y and is known as the relative metric initiated on Y by d. The space (Y, d_Y) is known as the metric subspace of the metric space (x,d).

Example 1 Consider the real line $(\mathbb{R}, u)$ with usual metric u given by $u(x,y) = |x - y| \;\; \forall\; x,y \in \mathbb{R}$ and the complex plane $(\mathbb{C}, d)$ with usual metric d given by $d(z,z') = |z - z'| \qquad \forall\, z,z' \in \mathbb{C}$

From definition of u and d, it is clear that

$u(x,y) = d(x,y) \qquad \forall\, x,y \in \mathbb{R}$ i.e., $u = d|_{\mathbb{R}}$

Therefore $(\mathbb{R}, u)$ is a metric subspace of the complex plane $(\mathbb{C}, d)$.

Example 2 Any subset of $\mathbb{R}$ (for eg. $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{N}$, $\mathbb{Z}$, $[0,1]$, $[0,1)$, $(0,1)$, $(0,1) \cup (2,3)$ etc.) is a metric subspace of the real line $(\mathbb{R}, u)$.

Definition 2 (Superspace of a Metric Space)

If y is a metric space and x is a subset of Y, we can induce metric on x by restricting the metric of Y on x. The question arises, can we do the reverse thing?

Suppose (x,d) is a metric space and Y is a proper superset of x. Can we define a metric on Y that is an extension of d? The answer is yes and it can be done in several ways, but we shall elaborate only one method.

Consider a metric space (x,d) and $Y \supsetneq X$. Since Y\x is non-empty, take any metric e on Y\x. Choose and fix two points $a \in X$ and $b \in Y \setminus X$.

Now define $D : Y \times Y \to \mathbb{R}$ as

$$D(x,y) = \begin{cases} d(x,y) & \text{if } x,y \in X \\ e(x,y) & \text{if } x,y \in Y \setminus X \\ d(x,a) + 1 + e(b,y) & \text{if } x \in X \text{ and } y \in Y \setminus X \\ e(x,b) + 1 + d(a,y) & \text{if } x \in Y \setminus X \text{ and } y \in X \end{cases},$$

Then D is a metric on Y such that $d = D|_{X \times X}$.

REFERENCES

1. E.T. Copson (2008). Metric Spaces, Cambridge University Press, Cambridge.
2. George F. Simmons (2003). Introduction to Topology and Modern Analysis, McGraw-Hill Book Co., New York.
3. M.K Singal and Asha Rani Singal (2010). Topics in Analysis II (Metric Spaces), R. Chand and Co., New Delhi.
4. Marcus, M. and Minc, H. (2004). A Survey of Matrix Theory and Matrix Inequalities, Allyn and Bacon, Boston, MA.
5. Micheal O. Searcoid (2012). Metric Spaces, Springer International Edition, New Delhi.
6. Royden, H. L. (2007). Real Analysis, 2nd edition, Macmillan, New York.
7. Satish Shirali and Harkrishan L. Vasudeva (2006). Metric Spaces, Springer.

Corresponding Author

Versha Chopra*