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A Study on Fractal Geometry & Its Measures and Dimensions

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Abstract – The aim of this chapter is to present necessary background material from the areas of fractal geometry, symbolic dynamics and renewal theory. To that end, we state various well-known results and examples, but also give some new results. With these foundations and those of Chapter 3, we will be able to achieve our overall goal of constructing and developing a theory of noncommutative fractal geometry.

Keywords:- Symbolic Dynamics, Fractal Geometry, Noncommutative, Symbolic Dynamics etc.

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INTRODUCTION

FRACTAL GEOMETRY

Let us begin by collecting relevant results from fractal geometry. The majority of the material detailed here is well-known and so is stated here without proof, with the exception of the final two results of Subsection 2.1.3 which do not seem to appear in the current literature. For the interested reader, there is an extensive literature available, with good overviews contained in [Fal1, Fal3, Man4, Pol2].

FRACTAL MEASURES AND DIMENSIONS

In the foundational essay [Man3], Mandelbrot introduced the subject of fractal geometry. One of the main motivations was to introduce tools which would be able to deal with irregular and fragmented patterns which occur in nature and science. Often, unlike \smooth" objects whose structure becomes simpler on a shrinking scale, fractal objects tend to be irregular or fragmented on a shrinking scale. Therefore, fractal sets are too irregular to be described either locally or globally with traditional geometric tools. Various attempts have been made to give a mathematically precise definition of a fractal, but in general such definitions have proven to be unsatisfactory. Therefore, it is often the case that a set is described as being fractal if it satisfies certain characteristics, for instance the above-described irregularity at all scales. Another characteristic is having a non-integer Hausdor fidimension, which is obtained from the Hausdorfi measure, where the Hausdor fi measure is defined in an analogous way to the n-dimensional Lebesgue measure, for n 2 N. In what follows, let n 2 N be fix.

Let E denote a subset of Rn, let s > 0 and let fi> 0. Define

$$\mathcal{H}^s_\eta(E) := \inf\left\{\sum_{k\in\mathbb{N}}|E_k|^s : E \subseteq \bigcup_{k\in\mathbb{N}} E_k \text{ and } |E_k| < \eta\right\}$$

to be the ø approximation to the s-dimensional Hausdor measure and deøne

$$\mathcal{H}^s(E) := \sup_{\eta>0} \mathcal{H}^s_\eta(E)$$

The following theorem gives a simple way of calculating the Hausdorfi dimension of self-similar sets satisfying the strong separation condition. In fact, the theorem holds under a slightly weaker condition, namely, the open set condition It is well-known that any compact totally disMeasurementected subset E of R with no isolated points is homeomorphic to the middle third Cantor set (see Corollary 30>4 of [Wil]). Therefore, we can view E in terms of its complement, that is, as a family of disjoint open intervals fIk > k 2 Ng. When viewing E in this way, it will always be assumed that the complementary intervals Ik are ordered so that their lengths are non-increasing. If, in addition, one imposes a certain porosity condition on

E, then one obtains bounds on the rate of decrease of the lengths of the complementary intervals and that the Hausdorø dimension must be strictly positive. The porosity condition with which we shall be concerned (especially in Subsection 4.1.2) is the following. We show how multifractal properties of a Borel probability measure ø supported on a non-empty compact fractal set E of R satisfying a certain porosity condition1 can be expressed in terms of the complementary intervals of the support of ø (by a fractal set we mean a non-

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empty totally dis Measurementected space with no isolated points). This allows the development of a noncommutative analogue of a coarse multifractal formalism for Measurementes' spectral triple representation of the set E. Speciøcally, we prove that from this new development one can recover the coarse multifractal box-counting dimension of ø. For a self-similar measure ø, given by an iterated function system S, we then show that our noncommutative coarse multifractal formalism gives rise to a noncommutative integral which recovers the associated self-similar multifractal measure ø, and we establish a relationship between the volume of such a noncommutative integral and the measure theoretical entropy of ø with respect to S. By reøning the methods of Antonescu-Ivan and Christensen given in [AIC1], we derive a (1; +)-summable spectral triple for each one-sided topologically exact subshift of ønite type (ø1A ; ø) equipped with an equilibrium measure øø (where ø 2 C(ø1A ;C) denotes some Holder continuous non-arithmetic potential function). We show that a variety of geometric and measure theoretic information can be recovered form such a spectral triple. We prove that Measurementes' pseudo-metric, given by our spectral triple, is a metric on the state space S(C(ø1A ;C)) of the Cø-algebra of complexvalued continuous functions deøned on ø1A , and that the topology induced by this metric is equivalent to the weakø-topology on S(C(ø1A ;C)). We show that the noncommutative integration theory of our spectral triple is capable of recovering the measure øø and that the noncommutative volume is equal to the reciprocal of the measure theoretical entropy of øø with respect to the left shift ø.

MOTIVATION AND HISTORY

In the **1980**s Measurementes formalised the notion of noncommutative geometry (see for instance [Con3, Con1]) and, in doing so, showed that the tools of differential geometry can be extended to certain non-Hausdor spaces known as bad quotients" and to spaces of a \fractal" nature. Such spaces are abundant in nature and commonly arise from various dynamical systems. A main idea of noncommutative geometry is to analyse geometric spaces using operator algebras, particularly Cfi-algebras. This idea first appeared in the work of Gelfand and Nafifimark [GN], where it was shown that a Cfi-algebra can be seen as the noncommutative analogue of the space of complex-valued continuous functions on a locally compact metric space Also, note that for a smooth compact spin Riemannian manifold, one can recover its smooth structure, its volume and its Riemannian metric directly from its standard Dirac operator (see [Jos]). Motivated by these observations, Measurementes proposed the concept of a spectral triple. A spectral triple is a triple (A;H;D) consisting of a Cfi-algebra A, which acts faithfully on a separable Hilbert space H, and an essentially self-adjoint unbounded operator D defined on H with compact resolvent such that the set

$$\{a \in A \,:\, \text{the operator } [D, \pi(a)] \text{ extends to a bounded operator defined on } H\}$$

is dense in A. (Here fi > A ! B(H) denotes the faithful action of A on H.) Measurementes showed that with such a structure one can obtain a pseudo-metric on the state space S(A) of A, analogous to how the Monge-Kantorovitch metric is defined on the space of probability measures on a compact metric space. In **1998** Riefiel [Rie2] and Pavlovific [Pav] established conditions under which Measurementes' pseudo-metric is a metric and established conditions under which the metric topology of Measurementes' pseudo-metric is equivalent to the weakfi-topology defined on S(A). Also, Measurementes [Con3] showed that a notion of dimension (called the metric dimension) and that a theory of integration can be derived for such structures. He also proved that for an arbitrary smooth compact spin Riemannian manifold there exists a spectral triple from which the metrical information, the measure theoretical information and the smooth structure of the manifold can be recovered (see [Con3, Ren]). This illustrates that a spectral triple allows one to move beyond the limits of classical Riemannian geometry. That is to say, not only is one able to recover classical aspects of Riemannian geometry, but through the notion of a spectral triple one is able to extend the tools of Riemannian geometry to situations that present themselves at the boundary of classically defined objects, for instance, objects which live" on the boundary of Teichmuller space (such as the noncommutative torus) or those of a fractal" nature (such as the middle third Cantor set). Although one of the original motivations for noncommutative geometry was to be able to deal with non-Hausdorfi spaces, such as foliated manifolds, which are often best represented by a noncommutative Cfi-algebra (see [Con3, Vfiar, Mar, Rie3]), this new theory has scope, even when the Cfi-algebra is commutative.

ANALYSIS OF THE STUDY

The material contained in this subsection forms the final section of the paper by Falconer and Samuel [FS]. Our main aim is to show how certain coarse multifractal information of a measure supported on a compact \fractal" subset of [0; 1] satisfying a porosity condition can be rediscovered through Connes' spectral triple, as given in Proposition 4.1.2. Recall that we let E denote a strongly porous compact totally disconnected subset of R with no isolated points, where we assume, without loss of generality, that f0; 1g ø E ø [0; 1]. Further, recall that we let flk >= (bøk ; b+k) > k 2 Ng denote the set of complementary intervals of E of finite length, ordered so that jlkj > jlk+1j.

for a given q 2 R the critical value

$$\inf\{p \in \mathbb{R} \,:\, Q^q |D|^{-p} \in \mathcal{L}^{1,+}(H)\}$$

reects the behaviour of the multifractal moment sums of E, given that E is strongly porous and that ø satisøes the mild density condition given in Equation (4.10). Moreover, from this result, in Corollary 4.1.10,

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we show that the noncommutative integral gives rise to a non-degenerate integral with respect to the underlying measure.

METHODS AND METHODOLOGY

In order to show that U = v, we use a scaling argument to show that U and v agree on a semi-ring which generates the Borel v-algebra. Then by an application of the Hahn-Kolmogorov Theorem (Theorem 3.3.5) the result follows. To this end, consider (i1; i2; > > > ; ik) 2 f1; 2; > > > ;mgk and let I >= si1si2 > > > sik (E). Then the singular values of v(vI)Qq v;vjDjøb(q) are precisely those corresponding to the complementary intervals contained in I and those corresponding to the complementary intervals whose closure intersects the boundary of I. Next, note that the following hold.

1. The mapping si1si2 > > > sik gives a bijection between the sets fIk > k 2 Ng and fIk > k N and Ik ø Ig.

2. For any interval J ø [0; 1], we have jsi1si2 > > > sik (J)j = ri1ri2 > > > rik jJj.

3. For any interval J ø [0; 1] of suøciently small diameter, we have that ø(si1si2 > > > sik (J)) pi1pi2 > > > pikø(J).

4. The Dixmier trace is linear and vanishes on operators with ønite dimensional range.

Observe that the set C(ø1A ;C) equipped with the supremum norm is a Cø-algebra, that L2(ø1A ; B; øø) is a complex Hilbert space and that (ø;L2(ø1A ; B; øø)) is a faithful ø-representation of C(ø1A ;C). Further, we have seen that Døø is a well deøned unbounded operator. Next, observe that the kernel of Døø consists of all equivalence classes of L2(ø1A ; B; øø) which contain some constant function on ø1A . Moreover, by the properties of a Gibbs measure, we have that Dø1 øø is a bounded operator on the complex Hilbert space ker(Døø)? ø L2(ø1A ; B; øø). Hence, the operator (1+D2 øø)ø1=2 is a bounded operator which can be approximated by operators in B(ker(Døø)?) with ønite dimensional ra ge. Therefore, Døø has a compact resolvent. Moreover, the sets Ran(Døøøi1) are L2-norm-dense in L2(ø1A ; B; øø). This follows, since the set of locally constant functions is L2- norm-dense in L2(ø1A ; B; øø), since the operator (Døøøi1) is linear and since we have the following.

The Dixmier Ideal and The Dixmier Trace

Here we give a complete proof (of our own design) of the fact that for a complex separable Hilbert space H, the Dixmier ideal L1;+(H) is an ideal of B(H) and that the Dixmier trace is a singular trace deøned on L1;+(H). The original proof can be found in [Dix1]. Before proving the main results we give several eigenvalue inequalities which will be required. Further, throughout this section, we let H denote a complex separable Hilbert space.

Lemma A.3.1. For each T 2 K(H) and each k 2 N, we have that øk(T) = øk(Tø). (Recall that øk(T) denotes the k-th largest singular value of T, where k 2 N.)

Proof. If h1 2 H is a non-zero eigenvector of TøT with eigenvalue z1 , then TøT(h1)øz1h1 = 0, and so TTøT(h1)øz1T(h1) = 0. Therefore, T(h1) is a non-zero eigenvector of TTø with the eigenvalue z1 . Similarly, if h2 is an eigenvector of TTø with eigenvalue z2 , then Tø(h2) is an eigenvector of TøT with eigenvalue z2 . Further, if h3 and h4 are two non-zero orthogonal eigenvectors of TøT with non-zero eigenvalue z3 , then we have that

$$\langle T(h_3), T(h_4)\rangle = \langle T^*T(h_3), h_4\rangle = z_3\langle h_3, h_4\rangle = 0.$$

Thus, TøT and TTø have the same eigenvalues with the same multiplicity. Note that we have implicitly used the assumption that T is compact, since we have used the fact that the eigen space of an eigen value of a compact operator is ønite dimensional.

CONCLUSION

In this thesis examples of spectral triples, which represent fractal sets, are examined and new insights into their noncommutative geometries are obtained.

Firstly, starting with Connes' spectral triple for a non-empty compact totally disconnected subset E of R with no isolated points, we develop a noncommutative coarse multifractal formalism. Specifically, we show how multifractal properties of a measure supported on E can be expressed in terms of a spectral triple and the Dixmier trace of certain operators. If E satisøes a given porosity condition, then we prove that the coarse multifractal box-counting dimension can be recovered. We show that for a self-similar measure ø, given by an iterated function system S deøned on a compact subset of R satisfying the strong separation condition, our noncommutative coarse multifractal formalism gives rise to a noncommutative integral which recovers the self-similar multifractal measure ø associated to ø, and we establish a relationship between the noncommutative volume of such a noncommutative integral and the measure theoretical entropy of ø with respect to S. Secondly, motivated by the results of Antonescu-Ivan and Christensen, we construct a family of (1; +)-summable spectral triples for a one-sided topologically exact subshift of ønite type (øN A ; ø). These spectral triples are constructed using equilibrium measures obtained from the Perron-Frobenius-Ruelle operator, whose potential function is non-arithemetic and Holder continuous. We show that the Connes' pseudo-metric, given by any one of these spectral triples, is a metric and that the metric topology agrees with the weakø-topology on the state space S(C(øN A);C). For each equilibrium measure øø we show that the noncommuative volume of the associated spectral triple is equal to the reciprocal of the measure theoretical entropy of øø with respect to the left shift ø (where it is assumed, without loss of generality, that the pressure of the potential function is equal to zero). We also show that the measure øø can be fully recovered from the noncommutative integration theory.

OUTLINE AND STATEMENT OF MAIN RESULTS

The main contributions of this thesis are contained. where our core results are contained in we give new results which are both interesting themselves and essential to the proofs of our main results. Below, we give a more detailed outline of the work carried out in this thesis. In this chapter, we begin by discussing some of the basic aspects of fractal geometry that will be required in the subsequent chapters. The _rst section, Section 2.1, is split into three main parts. A general and brief introduction to fractal measures and dimensions (Subsection 2.1.1), a brief review of the Minkowski content of a subset of R (Subsection 2.1.2) and _nally an introduction to the notions of coarse multifractal analysis (Subsection 2.1.3). The material contained in Subsection 2.1.1 and Subsection 2.1.2 is standard in the theory of fractal geometry and these subsections are respectively based on material contained in [Fal1] and [Fal2]. In Subsection 2.1.3, we de_ne the coarse multifractal box-counting dimension b(q) at q 2 R for a given Borel probability measure _ with compact support, where we use the extension for negative q introduced by Riedi [Rie1]. We then prove that an equivalent de_nition of b exists in terms of the complement of the support of _, provided that the support of _ is strongly porous In the next, we introduce the concept of a one-sided subshift of _nite type. We describe the thermodynamic formalism for this setting, as developed by Bowen and Ruelle ([Bow1, Bow2, Rue1, Rue2]). We state the results which give the existence of a Gibbs measure and the existence and uniqueness of an equilibrium measure on a one-sided topologically exact subshift of _nite type. Finally, in Theorem 2.2.10, a new notion of Haa basis for the Hilbert space L2(_1A ; B; _) is developed. (Here, (_1A ; _) denotes a one-sided topologically exact subshift of _nite type and _ denotes a Gibbs measure with support equal to _1A .) This concept enables us to describe in a natural way the _ltration on L2(_1A ; B; _) induced by the Gelfand-Na__mark-Segal completion and the AF-structure of the C_-algebra of complex-valued continuous functions de_ned on _1A . Thus, we are able to re_ne and develop the spectral triple of Antonescu-Ivan and Christensen's for an AF C_-algebra, in the setting of a one-sided topologically exact subshift of _nite type.

These counting results, interesting in themselves, also allow us to prove new results. In particular, they allow us to formulate a link between the notion of measure theoretical entropy and the notion of a noncommutative volume for a one-sided topologically exact subshift of _nite type equipped with an equilibrium measure.

To conclude the topic, we give three basic examples of spectral triples, examining their noncommutative geometries. Although most of the material in this section is well-known, it is often the case that many of the _ner details do not seem to appear in the literature. When this is the case we provide a full account. Speci_cally, we examine the noncommutative geometries of spectral triple representations of the following: the unit circle (Subsection 3.3.1), noncommutative tori (Subsection 3.3.2) and duals of countably in_nite discrete groups (Subsection 3.3.3). In the case of the noncommutative torus we take a more dynamical approach than that usually presented in the literature (see [Con3, V_ar, FGBV]). The material contained in Subsection 3.3.3 is based on material contained in [Con2] we focus on the case where E denotes a self-similar subset of R generated by an iterated function system of similarities S which satis_es the strong separation condition and where _ denotes a self-similar Borel probability measure on E. Here, we show that one can obtain a noncommutative integral which recovers the associated selfsimilar multifractal measure _. Before giving the statement of our result, we set the following notation

we focus on the case where E denotes a self-similar subset of R generated by an iterated function system of similarities S which satis_es the strong separation condition and where _ denotes a self-similar Borel probability measure on E. Here, we show that one can obtain a noncommutative integral which recovers the associated selfsimilar multifractal measure _. Before giving the statement of our result, we set the following notation.

1. Let (A;H;D) denote Connes' spectral triple for the set E where _ : A ! B(H) denotes the faithful action of A on H (note that A := C(E;C)).

2. Since S satis_es the strong separation condition, this implies that the set E is strongly porous. Letting _ denote the porosity constant of E, for each _ > _, let Q_;_ : H ! H denote the bounded linear operator as in Equation (1.1).

3. For a given limiting procedure W, let TrW denote the Dixmier trace with respect to W. Note that it is through the Dixmier trace that one obtains a noncommutative integral.

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